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Original Research Article

Theoretical Analysis of the Exponentiated Transmuted Kumaraswamy Distribution with Application

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Abstract

This research introduces a new four-parameter probability model which represents another generalization of the Transmuted kumaraswamy distribution. The developed model is named Exponentiated Transmuted Kumaraswamy Distribution (ETKD) and its basic statistical properties are studied and provided. An explicit expression for its moment, moment generating function, odds function, hazard function, survival function and quantile function were discussed and derived. The model parameters are estimated by using the technique of maximum likelihood. Based on the statistical significance of the model adequacy, the ETK distribution provides a better fit than the well-known *Transmuted Kumaraswamy* and Kumaraswamy distribution on the flood dataset.

Keywords: Generalization, Transmuted kumaraswamy distribution, Maximum likelihood estimation and model adequacy.

1.0 Introduction

The Kumaraswamy probability distribution was developed by Kumaraswamy (1980) which has a wider application in hydrology and related areas. Despite its similarity with Beta distribution, it is considered to be used best in terms of simulation studies because of its simple closed-form of both its probability density function and cumulative distribution function.

Based on the work of Nadarajah (2008), kumaraswamy distribution has been used in numerous researches in the context of hydrological literature because it is considered to serve as a "better substitute" to the beta distribution due to its simpler form nature, (see, Koutsoyiannis and Xanthopoulos (1989)). In numerous areas such as hydrology, Kumaraswamy distribution has received considerable interest, see Sundar and Subbiah (1989), Seifi et al. (2000), Ponnambalam et al. (2001) and Ganji et al. (2006).

Muhammad *et al.,(*2016) proposed a new three-parameter probability model named Transmuted Kumaraswamy distribution and derived its basic statistical properties. The proposed model was found to outperform some existing baseline distributions (models) when applied to real-life datasets. Mohammed and Abdullahi (2017), the study introduces a four-parameter probability model obtained by adding a shape parameter to New Weighted Exponential distribution. The two-parameter New Weighted Exponential distribution was generalized to its four-parameter variant entitled Kumaraswamy-New Weighted

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Exponential Distribution (K-NWED). Mathematical expressions for its Moments, Moment Generating Function (MGF), Reliability analysis, Limiting behavior and the Quantile Function of the K-NWED were presented. The parameters of the K-NWED were estimated using the technique of maximum likelihood. Cordeiro and de Castro (2009) proposed another family of the Kumaraswamy generalized distributions denoted by the (Kw–G). They derived and studied almost all the structural and statistical properties of the Kw–G distribution. Kumaraswamy distribution has received great attention over the years, (see Cordeiro et al. (2010, 2012), Jones (2009)) and Yahaya and Mohammed (2017).

Because of the flexibility of Kumaraswamy distribution, many kinds of research in the hydrological analysis were carried out. Gupta and Kundu (1999) introduce the exponentiated exponential distribution which considered to be a generalization form of the standard exponential distribution. Based on the same concept, four more different exponentiated type of distributions were proposed and the structural properties of the standard Gumbel, standard gamma, standard Weibull and standard Fréchet distributions were also provided by (Nadarajah and Kotz (2006)).

2.0 The Exponentiated Transmuted Kumaraswamy Distribution (ETKD)

The derivation of the Exponentiated family of distribution is defined by raising the cumulative distribution function of an arbitrary baseline distribution by a shape parameter say; $\gamma > 0$. Its cumulative density function (cdf) is given as;

$$F(x) = \left[G(x)\right]^{\gamma} \tag{1}$$

and its corresponding probability density function (pdf) is defined by;

$$f(x) = \gamma g(x)\left[G(x)\right]^{\gamma-1} \tag{2}$$

Now, to get the cdf and pdf of Exponentiated Transmuted Kumaraswamy Distribution (ETKD), we used equation (1) and (2). The pdf and cdf of Transmuted Kumaraswamy distribution are given by (3) and (4) respectively as

$$g(x) = \alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}\left\{1-\lambda+2\lambda(1-x^{\alpha})^{\theta}\right\} \tag{3}$$

$$G(x) = \left[1-(1-x^{\alpha})^{\theta}\right]\left[1+\lambda(1-x^{\alpha})^{\theta}\right] \tag{4}$$

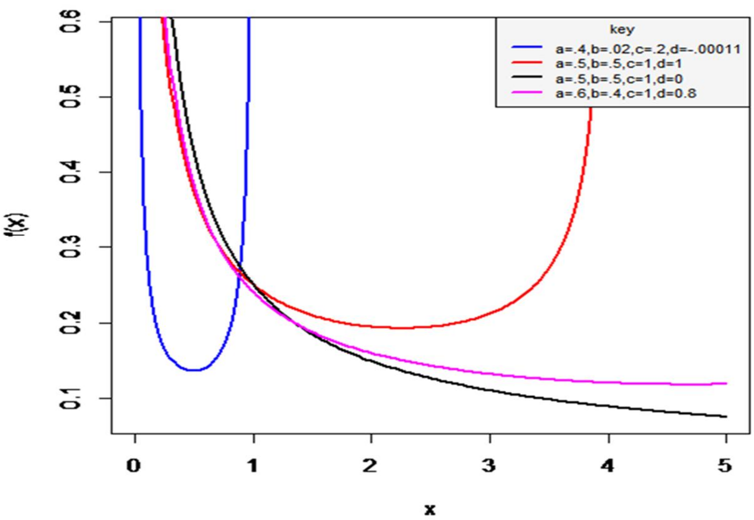

Figure 1: The probability density function of the ETKD for different values of the parameters

So,

$$F(x)=\left\{\left[1-(1-x^{\alpha})^{\theta}\right]\left[1+\lambda(1-x^{\alpha})^{\theta}\right]\right\}^{\gamma} \tag{5}$$

$$f(x)=\gamma\alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}\left\{1-\lambda+2\lambda(1-x^{\alpha})^{\theta}\right\}\left\{\left[1-(1-x^{\alpha})^{\theta}\right]\left[1+\lambda(1-x^{\alpha})^{\theta}\right]\right\}^{\gamma-1} \tag{6}$$

where, $\gamma\alpha\theta>0$ and $-1\le\lambda\le1$, $\gamma,\alpha\, and\, \theta$ are the shape parameters and λ the transmuting parameter.

By choosing some arbitrary values for parameters: $\gamma=a,\alpha=b,\theta=c,\ \lambda=d$ we provide some possible shape for the probability density function of ETKD as shown in Figure 1:

Equation (6) can be simplified further to give (7),
Since,

$$\left[1-(1-x^{\alpha})^{\theta}\right]^{\gamma-1}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma\gamma\Gamma(\theta i+1)}{i!\,j!\,\Gamma(\gamma-i)\Gamma(\theta i-j+1)}x^{\alpha j}$$

$$\left[1+\lambda(1-x^{\alpha})^{\theta}\right]^{\gamma-1}=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma\gamma\Gamma(\theta k+1)}{k!\,l!\,\Gamma(\gamma-k)\Gamma(\theta k-l+1)}\lambda^{k}x^{\alpha l}$$

Now,

$$f(x)=\gamma\alpha\theta\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\frac{\lambda^{k}\Gamma\gamma\Gamma(\theta i+1)\Gamma\gamma\Gamma(\theta k+1)}{i!\,j!\,k!\,l!\,\Gamma(\gamma-i)\Gamma(\theta i-j+1)\Gamma(\gamma-k)\Gamma(\theta k-l+1)}$$
$$x^{\alpha(1+j+l)-1}(1-x^{\alpha})^{\theta-1}\left\{1-\lambda+2\lambda(1-x^{\alpha})^{\theta}\right\}$$

$$f(x)=\gamma\alpha\theta m x^{\alpha(1+j+l)-1}(1-x^{\alpha})^{\theta-1}\left\{1-\lambda+2\lambda(1-x^{\alpha})^{\theta}\right\} \tag{7}$$

where,

$$m=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\frac{\lambda^{k}\Gamma\gamma\Gamma(\theta i+1)\Gamma\gamma\Gamma(\theta k+1)}{i!\,j!\,k!\,l!\,\Gamma(\gamma-i)\Gamma(\theta i-j+1)\Gamma(\gamma-k)\Gamma(\theta k-l+1)}$$

3.0 The rth Moment

Lemma 1: If X has the $ETK(x;\gamma,\alpha,\theta,\lambda)$ distribution, then the r*th* moment of X is given as follows;

$$E(x^{r})=\gamma\theta m(1-\lambda)\beta\left(1+j+l+\frac{r}{\alpha},\theta\right)+2\gamma\theta\lambda m\beta\left(1+j+l+\frac{r}{\alpha},2\theta\right)$$

$$m=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\frac{\lambda^{k}\Gamma\gamma\Gamma(\theta i+1)\Gamma\gamma\Gamma(\theta k+1)}{i!\,j!\,k!\,l!\,\Gamma(\gamma-i)\Gamma(\theta i-j+1)\Gamma(\gamma-k)\Gamma(\theta k-l+1)}$$

Proof: Let X have ETKD, then the r*th* moment of X is given by;

$$\mu_{r}^{'}=E(x^{r})=\int_{0}^{1}x^{r}f(x)dx$$

$$E(x^{r})=\gamma\alpha\theta m\int_{x=0}^{1}x^{\alpha(1+j+l)+r-1}(1-x^{\alpha})^{\theta-1}\left\{1-\lambda+2\lambda(1-x^{\alpha})^{\theta}\right\}dx$$

$$E(x^{r})=\gamma\alpha\theta m\left((1-\lambda)\int_{x=0}^{1}x^{\alpha(1+j+l)+r-1}(1-x^{\alpha})^{\theta-1}dx+2\lambda\int_{x=0}^{1}x^{\alpha(1+j+l)+r-1}(1-x^{\alpha})^{2\theta-1}dx\right)$$

$$E(x^r) = \gamma\alpha\theta m\left((1-\lambda)\int_{x=0}^{1} x^{\alpha(1+j+l+\frac{r}{\alpha})-1}(1-x^{\alpha})^{\theta-1}dx + 2\lambda\int_{x=0}^{1} x^{\alpha(1+j+l+\frac{r}{\alpha})-1}(1-x^{\alpha})^{2\theta-1}dx \right)$$

Let,

$$y = x^{\alpha}, x = y^{\frac{1}{\alpha}} \ and \ dx = \frac{1}{\alpha}y^{\frac{1}{\alpha}-1}dy$$

$$E(x^r) = \gamma\alpha\theta m\left((1-\lambda)\int_{x=0}^{1} y^{(1+j+l+\frac{r}{\alpha})}y^{-\frac{1}{\alpha}}(1-y)^{\theta-1}\frac{1}{\alpha}y^{\frac{1}{\alpha}-1}dy + 2\lambda\int_{x=0}^{1} y^{(1+j+l+\frac{r}{\alpha})}y^{-\frac{1}{\alpha}}(1-y)^{2\theta-1}\frac{1}{\alpha}y^{\frac{1}{\alpha}-1}dy \right)$$

$$E(x^r) = \gamma\theta m(1-\lambda)\beta\left(1+j+l+\frac{r}{\alpha},\theta\right) + 2\gamma\theta\lambda m\beta\left(1+j+l+\frac{r}{\alpha},2\theta\right)$$

$$E(x^r) = \gamma\theta m(1-\lambda)\Phi_{1,r} + 2\gamma\theta\lambda m\Phi_{2,r} \tag{8}$$

where,

$$\Phi_{z,r} = \beta\left(1+j+l+\frac{r}{\alpha},Z\theta\right), \qquad for\ z = 1,2$$

3.1 *Raw Moments about the Origin*

Putting r = 1, 2, 3 and 4 in (7) first four raw moments are

$$E(x^1) = \gamma\theta m(1-\lambda)\beta\left(1+j+l+\frac{1}{\alpha},\theta\right) + 2\gamma\theta\lambda m\beta\left(1+j+l+\frac{1}{\alpha},2\theta\right) \tag{9}$$

$$E(x^2) = \gamma\theta m(1-\lambda)\beta\left(1+j+l+\frac{2}{\alpha},\theta\right) + 2\gamma\theta\lambda m\beta\left(1+j+l+\frac{2}{\alpha},2\theta\right) \tag{10}$$

$$E(x^3) = \gamma\theta m(1-\lambda)\beta\left(1+j+l+\frac{3}{\alpha},\theta\right) + 2\gamma\theta\lambda m\beta\left(1+j+l+\frac{3}{\alpha},2\theta\right) \tag{11}$$

$$E(x^4) = \gamma\theta m(1-\lambda)\beta\left(1+j+l+\frac{4}{\alpha},\theta\right) + 2\gamma\theta\lambda m\beta\left(1+j+l+\frac{4}{\alpha},2\theta\right) \tag{12}$$

The expressions for variance is given by;

$$\text{var}(X) = E(X^2) - \left(E(X)\right)^2$$

$$\begin{aligned} V\text{ar}(X) = {} & \gamma\theta m(1-\lambda)\beta\left(1+j+l+\frac{2}{\alpha},\theta\right) + 2\gamma\theta\lambda m\beta\left(1+j+l+\frac{2}{\alpha},2\theta\right) \\ & -\left(\gamma\theta m(1-\lambda)\beta\left(1+j+l+\frac{1}{\alpha},\theta\right) + 2\gamma\theta\lambda m\beta\left(1+j+l+\frac{1}{\alpha},2\theta\right)\right)^2 \end{aligned} \tag{13}$$

The coefficient of variation (CV), skewness and kurtosis measures can now be calculated using the following relationships

$$CV = \frac{\sqrt{\text{var}(X)}}{E(X)}$$

$$\text{Skewness(X)} = \frac{E(X-E(X))^3}{\text{var}(X)^{\frac{3}{2}}}$$

$$\text{Kurtosis(X)}=\frac{E(X-E(X))^4}{\text{var}(X)^2}$$

The above three statistics can be obtained using equations (7).

3.2 *Moment Generating Function*

Lemma 2: If X has the $ETK(x;\gamma,\alpha,\theta,\lambda)$ distribution, then the Moment Generating Function of X is given as follows;

$$M_x(t)=\gamma\theta m\sum_{n=0}^{\infty}\frac{t^n}{n!}\left((1-\lambda)\beta\left(1+j+l+\frac{n}{\alpha},\theta\right)+2\lambda\beta\left(1+j+l+\frac{n}{\alpha},2\theta\right)\right) \quad (14)$$

Where, $m=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\frac{\lambda^k\Gamma\gamma\Gamma(\theta i+1)\Gamma\gamma\Gamma(\theta k+1)}{i!j!k!l!\Gamma(\gamma-i)\Gamma(\theta i-j+1)\Gamma(\gamma-k)\Gamma(\theta k-l+1)}$

Proof: Let X have ETKD, then the Moment Generating Function of X is given as;

$$M_x(t)=\int_{x=0}^{1}e^{tx}f(x)dx$$

Using the Taylor series, the function e^{tx} becomes;

$$e^{tx}=\sum_{n=0}^{\infty}\frac{t^nx^n}{n!}$$

$$M_x(t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_{x=0}^{1}x^nf(x)dx$$

$$M_x(t)=\gamma\alpha\theta m\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_{x=0}^{1}x^{\alpha(1+j+l)+n-1}(1-x^\alpha)^{\theta-1}\left\{1-\lambda+2\lambda(1-x^\alpha)^\theta\right\}dx$$

$$M_x(t)=\gamma\alpha\theta m\sum_{n=0}^{\infty}\frac{t^n}{n!}\left((1-\lambda)\int_{x=0}^{1}x^{\alpha(1+j+l+\frac{r}{\alpha})-1}(1-x^\alpha)^{\theta-1}dx+2\lambda\int_{x=0}^{1}x^{\alpha(1+j+l+\frac{r}{\alpha})-1}(1-x^\alpha)^{2\theta-1}dx\right)$$

$$M_x(t)=\gamma\alpha\theta m\sum_{n=0}^{\infty}\frac{t^n}{n!}\left((1-\lambda)\int_{x=0}^{1}x^{\alpha(1+j+l+\frac{r}{\alpha})-1}(1-x^\alpha)^{\theta-1}dx+2\lambda\int_{x=0}^{1}x^{\alpha(1+j+l+\frac{r}{\alpha})-1}(1-x^\alpha)^{2\theta-1}dx\right)$$

Let,

$$y=x^\alpha,x=y^{\frac{1}{\alpha}}\ and\ dx=\frac{1}{\alpha}y^{\frac{1}{\alpha}-1}dy$$

$$M_x(t)=\gamma\alpha\theta m\sum_{n=0}^{\infty}\frac{t^n}{n!}\left((1-\lambda)\int_{x=0}^{1}y^{(1+j+l+\frac{n}{\alpha})}y^{-\frac{1}{\alpha}}(1-y)^{\theta-1}\frac{1}{\alpha}y^{\frac{1}{\alpha}-1}dy+2\lambda\int_{x=0}^{1}y^{(1+j+l+\frac{n}{\alpha})}y^{-\frac{1}{\alpha}}(1-y)^{2\theta-1}\frac{1}{\alpha}y^{\frac{1}{\alpha}-1}dy\right)$$

$$M_x(t)=\gamma\theta m\sum_{n=0}^{\infty}\frac{t^n}{n!}(1-\lambda)\beta\left(1+j+l+\frac{n}{\alpha},\theta\right)+2\gamma\theta\lambda m\sum_{n=0}^{\infty}\frac{t^n}{n!}\beta\left(1+j+l+\frac{n}{\alpha},2\theta\right)$$

3.3 *Quantile Function*

Lemma 3: If X has the $ETK(x;\gamma,\alpha,\theta,\lambda)$ distribution, then the Quantile Function of X is given as follows;

$$x_u = \left[1-\left(1-\frac{(1+\lambda)-\sqrt{(1+\lambda)^2-4\lambda u^{1/\alpha}}}{2\lambda}\right)^{1/\theta}\right]^{1/\alpha} \quad (15)$$

Proof: Let X have ETKD, then the Quantile Function of X is given as;

$$u = F(x)$$

$$u = \left\{\left[1-(1-x^{\alpha})^{\theta}\right]\left[1+\lambda(1-x^{\alpha})^{\theta}\right]\right\}^{\gamma}$$

$$u^{1/\gamma} = \left[1-(1-x^{\alpha})^{\theta}\right]\left[1+\lambda(1-x^{\alpha})^{\theta}\right]$$

$$u^{1/\gamma} = (1+\lambda)\left[1-(1-x^{\alpha})^{\theta}\right]-\lambda\left[1-(1-x^{\alpha})^{\theta}\right]^2$$

$$\lambda\left[1-(1-x^{\alpha})^{\theta}\right]^2-(1+\lambda)\left[1-(1-x^{\alpha})^{\theta}\right]+u^{1/\gamma}=0$$

$$\left[1-(1-x^{\alpha})^{\theta}\right] = \frac{(1+\lambda)-\sqrt{(1+\lambda)^2-4\lambda u^{1/\alpha}}}{2\lambda}$$

$$(1-x^{\alpha})^{\theta} = 1-\frac{(1+\lambda)-\sqrt{(1+\lambda)^2-4\lambda u^{1/\alpha}}}{2\lambda}$$

$$x^{\alpha} = 1-\left(1-\frac{(1+\lambda)-\sqrt{(1+\lambda)^2-4\lambda u^{1/\alpha}}}{2\lambda}\right)^{1/\theta}$$

$$x_u = \left[1-\left(1-\frac{(1+\lambda)-\sqrt{(1+\lambda)^2-4\lambda u^{1/\alpha}}}{2\lambda}\right)^{1/\theta}\right]^{1/\alpha}$$

3.4 *Survival function*

Survival function is the probability that a system will survive beyond a given time. Mathematically, the survival function of ETKD is define by:

$$S(X;\gamma,\alpha,\theta,\lambda) = 1-F(x;\gamma,\alpha,\theta,\lambda)$$

$$S(X) = 1-\left(\left[1-(1-x^{\alpha})^{\theta}\right]\left[1+\lambda(1-x^{\alpha})^{\theta}\right]\right)^{\gamma} \quad (16)$$

By choosing some arbitrary values for parameters: $\gamma = a, \alpha = b, \theta = c,\ \lambda = d$ we provide some possible shape for the survival function of the ETKD as shown in Figure 2:

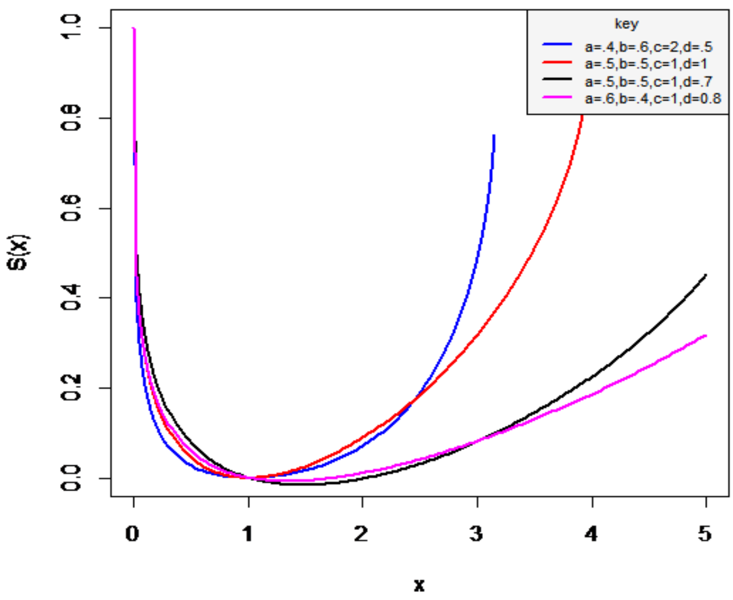

Figure 2: The plot of survival function of the ETKD for different values of the parameters

3.5 *Hazard function*

Hazard function is also called the failure or risk function, is the probability that a component will fail or die for an interval of time. The hazard function is define as;

$$h(x)=\frac{\gamma\alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}\left\{1-\lambda+2\lambda(1-x^{\alpha})^{\theta}\right\}\left\{\left[1-(1-x^{\alpha})^{\theta}\right]\left[1+\lambda(1-x^{\alpha})^{\theta}\right]\right\}^{\gamma-1}}{1-\left(\left[1-(1-x^{\alpha})^{\theta}\right]\left[1+\lambda(1-x^{\alpha})^{\theta}\right]\right)^{\gamma}} \quad (17)$$

3.6 *Odds Function*

Odds function for ETKD can be defined as:

$$O(x;\gamma,\alpha,\theta,\lambda)=\frac{F(x;\gamma,\alpha,\theta,\lambda)}{S(x;\gamma,\alpha,\theta,\lambda)}$$

$$O(x;\gamma,\alpha,\theta,\lambda)=\frac{\left(\left[1-(1-x^{\alpha})^{\theta}\right]\left[1+\lambda(1-x^{\alpha})^{\theta}\right]\right)^{\gamma}}{1-\left(\left[1-(1-x^{\alpha})^{\theta}\right]\left[1+\lambda(1-x^{\alpha})^{\theta}\right]\right)^{\gamma}} \quad (18)$$

3.7 *Estimation of Parameters of the ETKD*

The estimation of the parameters of the ETKD is done by using the method of maximum likelihood estimation. Let $X_1, X_2, \ldots\ldots, X_n$ be a random sample from the ETKD with unknown parameter vector $\Omega=(\gamma,\alpha,\theta,\lambda)^T$. The total log-likelihood function for Ω is obtained from *f(x)* as follow

$$\ln L(\Omega) = n\ln\gamma + n\ln\alpha + n\ln\theta + (\alpha-1)\sum_{i=1}^{n}\ln x_i + +(\theta-1)\sum_{i=1}^{n}\ln(1-x_i^{\alpha}) +$$
$$\sum_{i=1}^{n}\ln\left(1-\lambda+2\lambda(1-x_i^{\alpha})^{\theta}\right)+(\gamma-1)\sum_{i=1}^{n}\ln\left[1-(1-x_i^{\alpha})^{\theta}\right]+(\gamma-1)\sum_{i=1}^{n}\ln\left[1+\lambda(1-x_i^{\alpha})^{\theta}\right] \quad (19)$$

$$\frac{\delta \ln L(\Omega)}{\delta\lambda} = \sum_{i=1}^{n}\frac{2(1-x_i^{\alpha})^{\theta}-1}{\left(1-\lambda+2\lambda(1-x_i^{\alpha})^{\theta}\right)} + (\gamma-1)\sum_{i=1}^{n}\frac{(1-x_i^{\alpha})^{\theta}}{\left[1+\lambda(1-x_i^{\alpha})^{\theta}\right]} = 0$$

So, differentiating $\ln L(\Omega)$ partially with respect to each of the parameter $\Omega = (\gamma,\alpha,\theta,\lambda)^T$ and setting the results equal to zero gives the maximum likelihood estimate of the respective parameters. The partial derivatives of $\ln L(\Omega)$ with respect to each parameter is given by:

$$\frac{\delta \ln L(\Omega)}{\delta\gamma} = \frac{n}{\gamma} + \sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\ln\left[1-(1-x_i^{\alpha})^{\theta}\right] + \sum_{i=1}^{n}\ln\left[1+\lambda(1-x_i^{\alpha})^{\theta}\right] = 0$$

$$\hat{\gamma} = \frac{-n}{\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\ln\left[1-(1-x_i^{\alpha})^{\theta}\right] + \sum_{i=1}^{n}\ln\left[1+\lambda(1-x_i^{\alpha})^{\theta}\right]}$$

$$\frac{\delta \ln L(\Omega)}{\delta\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln x_i - (\theta-1)\sum_{i=1}^{n}\frac{x_i^{\alpha}\ln x_i}{(1-x_i^{\alpha})} - \sum_{i=1}^{n}\frac{2\lambda\theta(1-x_i^{\alpha})^{\theta-1}x_i^{\alpha}\ln x_i}{\left(1-\lambda+2\lambda(1-x_i^{\alpha})^{\theta}\right)}$$
$$-(\gamma-1)\sum_{i=1}^{n}\frac{\theta(1-x_i^{\alpha})^{\theta-1}x_i^{\alpha}\ln x_i}{\left[1-(1-x_i^{\alpha})^{\theta}\right]} - (\gamma-1)\sum_{i=1}^{n}\frac{\lambda\theta(1-x_i^{\alpha})^{\theta-1}x_i^{\alpha}\ln x_i}{\left[1+\lambda(1-x_i^{\alpha})^{\theta}\right]} = 0$$

$$\frac{\delta \ln L(\Omega)}{\delta\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\ln(1-x_i^{\alpha}) + \sum_{i=1}^{n}\frac{2\lambda(1-x_i^{\alpha})^{\theta}\ln(1-x_i^{\alpha})^{\theta}}{\left(1-\lambda+2\lambda(1-x_i^{\alpha})^{\theta}\right)} - (\gamma-1)\sum_{i=1}^{n}\frac{(1-x_i^{\alpha})^{\theta}\ln(1-x_i^{\alpha})^{\theta}}{\left[1-(1-x_i^{\alpha})^{\theta}\right]}$$
$$+(\gamma-1)\sum_{i=1}^{n}\frac{\lambda(1-x_i^{\alpha})^{\theta}\ln(1-x_i^{\alpha})^{\theta}}{\left[1+\lambda(1-x_i^{\alpha})^{\theta}\right]} = 0$$

$$\frac{\delta \ln L(\Omega)}{\delta\lambda} = \sum_{i=1}^{n}\frac{2(1-x_i^{\alpha})^{\theta}-1}{\left(1-\lambda+2\lambda(1-x_i^{\alpha})^{\theta}\right)} + (\gamma-1)\sum_{i=1}^{n}\frac{(1-x_i^{\alpha})^{\theta}}{\left[1+\lambda(1-x_i^{\alpha})^{\theta}\right]} = 0$$

Hence, the MLE is obtained by solving this nonlinear system of equations. Solving this system of nonlinear equations is complicated, we can therefore use statistical software to solve the equations numerically.

5.0 Application

This section presents the data analyses in order to assess the goodness-of-fit of the ETK distribution. The first data set is from (Dumonceaux and Antle, (1973)); with respect to the flood data with 20 observations and is given in the table below.

Table 1: Flood data

0.265, 0.269, 0.297, 0.315, 0.3235, 0.338, 0.379, 0.379, 0.392, 0.402, 0.412, 0.416, 0.418, 0.423, 0.449, 0.484, 0.494, 0.613, 0.654, 0.74.

In order to compare the flexibility of the distributions, I consider the Kolmogorov-Smirnov (K-S) test, Cramér-von Mises and Anderson-Darling goodness-of-fit statistics for the flood data. Table 2 gives the MLEs of the unknown parameters of the Exponentiated Transmuted Kumaraswamy (ETK), Transmuted Kumaraswamy (TK) and Kumaraswamy (K) distributions. These results shows that the ETK distribution provides an adequate fit for the flood data.

Table 2. Shows MLEs of the unknown Parameters for the flood data and the goodness-of-fit measures which includes K-S test, Cramér-von Mises and Anderson-Darling goodness-of-fit are given in the table below.

No.	Distribution	$\widehat{\alpha}$	$\widehat{\theta}$	$\widehat{\lambda}$	$\widehat{\gamma}$	$\boldsymbol{K-S}$ **Test**	**W**	**A**
1	ETK	1.7616	4.4083	0.8685	4.1777	0.17153	0.0835	0.5178
2	TK	3.7252	10.9575	0.6143	–	0.1930	0.1409	0.8408
3	K	3.3631	11.7882	-	–	0.2109	0.1658	0.9722

Table 2 indicates that all the three measures considered in this analysis i.e. Cramér-von Mises test statistics and Anderson-Darling goodness statistics have the smallest values for the *Exponentiated Transmuted Kumaraswamy* distribution for the flood data concerning the *Transmuted Kumaraswmy and Kumaraswmy* distributions. Based on these goodness-of-fit measures I conclude that the *Exponentiated Transmuted Kumaraswamy* distribution provides a better fit than the *TransmutedKumaraswmy and Kumaraswmy* distribution.

5.0 Conclusion

A shape parameter was added to the Transmuted Kumaraswamy Distribution in order to increase its flexibility. An explicit expression for some of its basic statistical properties were studied and derived. In terms of the statistical significance of the model adequacy, the ETK distribution leads to a better fit than the well-known TK and Kw distribution on the flood dataset.

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