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Original Research Article

Theoretical Study of Four-Parameter Odd-Generalized Exponential-Pareto Distribution

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Abstract

In this article, we introduce a new generalization of Pareto distribution entitled Odd Generalized Exponential-Pareto Distribution. An explicit derivation of some statistical properties of the proposed distribution including moments, moment generating function, quantile function, survival function, and hazard function was conducted. Some of the graphical plots generated based on some randomly selected parameter values indicate that the proposed distribution will be very suitable in analysing positively skewed datasets. The parameters of the distribution were estimated using the method of maximum likelihood.

Keywords: Generalized Exponential Distribution, Pareto Distribution, Quantile Function, Moments, Moment Generating Function.

1.0 Introduction

The Pareto distribution is a well-known probability distribution that is used in analyzing and modeling skewed data in different fields of human endeavor. It has been studied and applied by different researchers. Pickand (1975) was the first researcher to generalized Pareto distribution. The distribution was used to model prevalence of earthquakes, forest fire areas, oil and gas field sizes (Burroughs and Tebbens, 2001), as well as in researches related to online analytical processing (OLAP) meant to obtain meaningful information easily from a large amount of data residing in a data warehouse (Nadeau and Teorey, 2003).

Many lifetime models have been proposed by different researchers using different generalization procedures to analyze data for certain purposes like the flexibility of properties and performance of new distribution compared to the original distribution; however, there are still some real datasets that cannot be fitted with some existing probability distributions. To fill in this gap several researchers proposed some generalizations to the existing distributions. This notable effort creates an avenue for researchers to continue proposing and studying more new distributions.

Mohammed and Abdullahi (2017) proposed new distribution called Mathematical Study on Kumaraswamy New Weighted Distribution. The distribution is positively skewed and has at least one mode. Some of the basic properties of the developed model were presented.

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The available literature studied earlier highlighted that almost all generalized distributions (in which one, two, or more parameters are added) provided a better fit than their counterparts with less number of parameters. For instance, Mahmoudi (2011) introduced the Beta Generalized Pareto Distribution (BGPD) which served as an extension of the well-known Generalized Pareto Distribution (GPD) applied in modeling extreme value data due to its long tail feature. The study highlighted that, BGPD is more flexible due to its interesting properties and as such provided a better fit than those distributions with which it was compared.

Bourguignon et al. (2012) studied a new distribution called the Kumaraswany-Pareto distribution. An application to a real dataset showed that the fit of the new model is greater than the fit of its main sub-model- the Pareto distribution.

Merovci et al. (2014) proposed a new probability model called the transmuted Pareto distribution which extends the Pareto distribution in the analysis of data with real support. An application of the transmuted Pareto distribution to real data showed that the new distribution provided a better fit than the Pareto distribution.

El-Damcese et al. (2015) extended the Gompertz distribution to a four-parameter probability distribution called the Odd Generalized Exponential-Gompertz Distribution (OGEGD) by applying the OGE of family generator proposed by Tahir et al. (2015). The result showed that, OGEPD performed well or better than its sub-models (the generalized exponential and Gompertz distributions).

The Cumulative Distribution Function (CDF) of a Pareto random variable X with parameter θ and k is given by;

$$G(x,\theta,k) = 1 - \left(\frac{\theta}{x}\right)^k \qquad \theta \le x < \infty, k, \theta > 0 \tag{1}$$

While the corresponding Probability Density Function (PDF) is given by:

$$g(x,\theta,k) = \frac{k\theta^k}{x^{k+1}} \qquad \theta \le x < \infty, k, \theta > 0 \tag{2}$$

where $\theta > 0$ is a scale parameter and $k > 0$ is the shape parameter

Gupta and Kundu (1999) introduced the Generalized Exponential Distribution (GED) also known as Exponentiated Exponential Distribution (EED). It is a two-parameter probability distribution whose CDF of GED is given by:

$$G(x,\alpha,\lambda) = \left(1 - e^{-\lambda x}\right)^\alpha \qquad x, \alpha, \lambda > 0 \tag{3}$$

where α is the shape and λ is the scale parameter.

The corresponding PDF is:

$$g(x,\alpha,\lambda) = \alpha\lambda\left(1 - e^{-\lambda x}\right)^{\alpha-1} e^{-\lambda x} \qquad x, \alpha, \lambda > 0 \tag{4}$$

The Odd Generalized Exponential (OGE) family introduced by Tahir et al. (2015) is defined as follows; if the baseline distribution depends on a parameter vector ϕ having CDF, PDF and survival function denoted by $G(x;\phi)$, $g(x;\phi)$ and $\bar{G}(x;\phi) = 1 - G(x;\phi)$ respectively, then the CDF of the OGE family is obtained by replacing x in equation (3) with the odd ratio: $\frac{G(x;\phi)}{\bar{G}(x;\phi)}$.

Hence the following equation was obtained:

$$F(x;\alpha,\lambda,\phi)=\left(1-e^{-\lambda\frac{G(x;\phi)}{\bar{G}(x;\phi)}}\right)^{\alpha} \tag{5}$$

where $\alpha>0,\ \lambda>0,\phi>0$ is one additional parameters.

The corresponding PDF is:

$$f(x;\alpha,\lambda,\phi)=\frac{\lambda\alpha g(x;\phi)}{\bar{G}(x;\phi)^2}\mathrm{e}^{-\lambda\frac{G(x;\phi)}{\bar{G}(x;\phi)}}\left(1-\mathrm{e}^{-\lambda\frac{G(x;\phi)}{\bar{G}(x;\phi)}}\right)^{\alpha-1} \tag{6}$$

Hence, this article derived some statistical properties of the Four-Parameter Odd Generalized Exponential Pareto (OGE-P) Distribution.

This article is organised as follows: In section 2, we derived and expand the CDF and PDF of OGE-P distribution, in section 3, some of the statistical properties including moments, moment generating function, quantile function, median, and reliability analysis of the new distribution were derived. The parameters of the distribution were estimated using the method of maximum likelihood in section 4. Lastly the article is concluded in section 5.

2.0 The Four-Parameter OGE-P Distribution

The random variable X is said to follow an OGE-P distribution with parameters α,λ,θ and k if its CDF is given by:

$$F(x;\alpha,\lambda,\theta,k)=\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha}\qquad x\geq\theta,\ \ \alpha,\lambda,\theta,k>0 \tag{7}$$

The corresponding PDF is given by:

$$\boldsymbol{f(x;\alpha,\lambda,\theta,k)=\alpha\lambda k\theta^{-k}x^{k-1}e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha-1}} \tag{8}$$

2.1 *Some Useful Expansions for the CDF and PDF of Four-Parameter OGE-P Distribution*

Here, we provide some useful expansions for the CDF and PDF of OGE-P distribution by making use of the concept of generalized binomial expansion and power series expansion. This will aid in finding some other properties of the proposed distribution.
Now, consider the CDF provided in (7):

$$F(x;\alpha,\lambda,\theta,k)=\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha}$$

Using Binomial expansion for the term $\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha}$ we obtain:

$$F(x,\alpha,\lambda,\theta,k)=\sum_{i=0}^{\infty}\binom{\alpha}{i}(-1)^{i}e^{-i\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)} \tag{9}$$

and now using series expansion for the exponential term, we have:

$$e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)} = \sum_{j=0}^{\infty} \frac{(-1)^j\left(i\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)\right)^j}{j!} = \sum_{j=0}^{\infty} \frac{(-1)^j (i)^j (\lambda)^j \left(\left(\frac{\theta}{x}\right)^{-k}-1\right)^j}{j!} \qquad (10)$$

by substituting (10) into (9), we obtain:

$$F(x;\alpha,\lambda,\theta,k) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j}(i\lambda)^j \binom{\alpha}{i}\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)^j}{j!} \qquad (11)$$

Now, consider the PDF provided in (8):

$$f(x;\alpha,\lambda,\theta,k) = \alpha\lambda k\theta^{-k}x^{k-1}e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha-1} \qquad (12)$$

Since, $0 < 1 - e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)} < 1$, then using binomial expansion we obtain

$$\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha-1} = \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i e^{-i\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)} \qquad (13)$$

putting (13) in to (12), we have:

$$f(x;\alpha,\lambda,\theta,k) = \alpha\lambda k\theta^{-k}x^{k-1}\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\, e^{-\lambda(1+i)\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)} \qquad (14)$$

again using series expansion for the exponential term, we have:

$$e^{-\lambda(1+i)\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)} = \sum_{j=0}^{\infty}\frac{(-1)^j\left(\lambda(1+i)\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)\right)^j}{j!}$$

$$e^{-\lambda(1+i)\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)} = \sum_{j=0}^{\infty}\frac{(-1)^j(1+i)^j(\lambda)^j\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)^j}{j!} \qquad (15)$$

putting (15) in to (14), it becomes:

$$f(x;\alpha,\lambda,\theta,k) = \alpha k\theta^{-k}x^{k-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}(\lambda)^{1+j}(1+i)^j\binom{\alpha-1}{i}\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)^j}{j!} \qquad (16)$$

Equation (11) and (16) are the respective CDF and PDF of the OGE-P distribution in binomial and power series form.

2.3 *Survival and Hazard Functions*

The survival function of OGE-P distribution is given as;

$$S(x;\alpha,\lambda,\theta,k) = 1-\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha} \qquad x \ge \theta,\ \ \alpha,\lambda,\theta,k>0 \tag{17}$$

Using binomial and power series expansion, we write survival function as:

$$S(x;\alpha,\lambda,\theta,k) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}(i\lambda)^{j}\binom{\alpha}{i}\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)^{j}}{j!} \tag{18}$$

The hazard function of OGE-P distribution is defined as:

$$h(x;\alpha,\lambda,\theta,k) = \frac{\alpha\lambda k\theta^{-k}x^{k-1}e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha-1}}{1-\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha}}\ x \ge \theta,\ \ \alpha,\lambda,\theta,k>0 \tag{19}$$

The plot of the CDF, PDF, survival function and hazard function for different values of the parameters are provided in figures below, where $a=\alpha$, $b=\lambda$, $c=\theta$ and $d=k$.

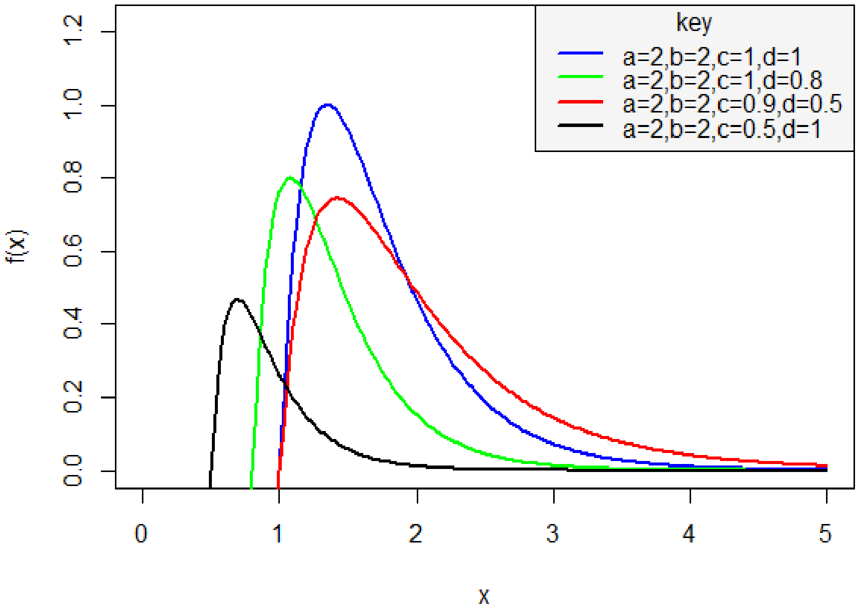

Figure 1: The PDF plot of OGE-P distribution

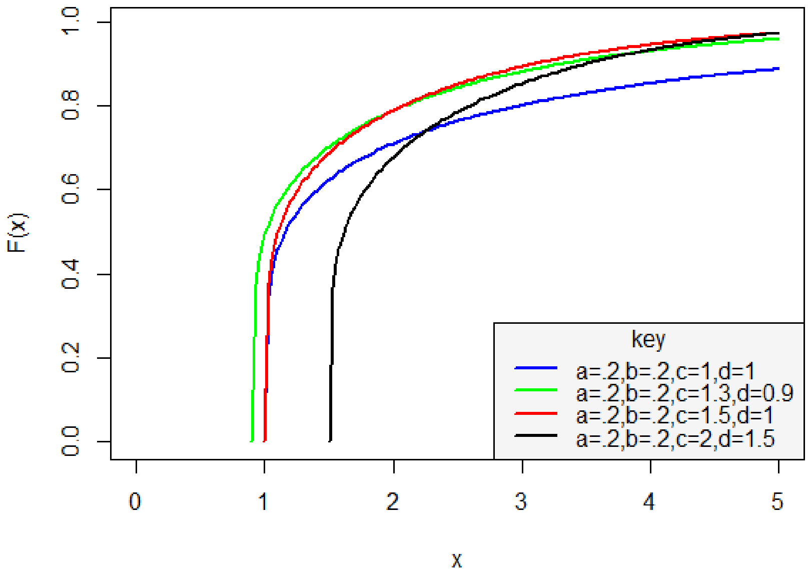

Figure 2: The CDF plot of OGE-P distribution

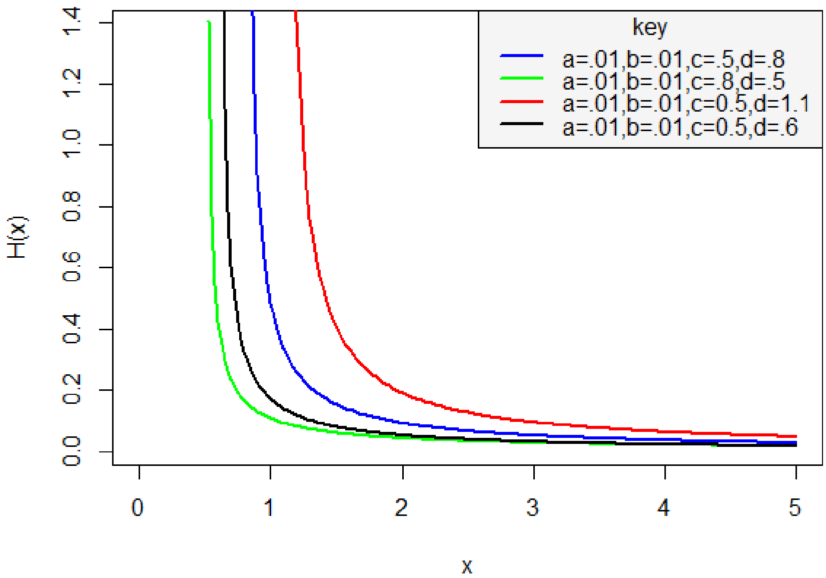

Figure 3: The Hazard plot of OGE-P distribution

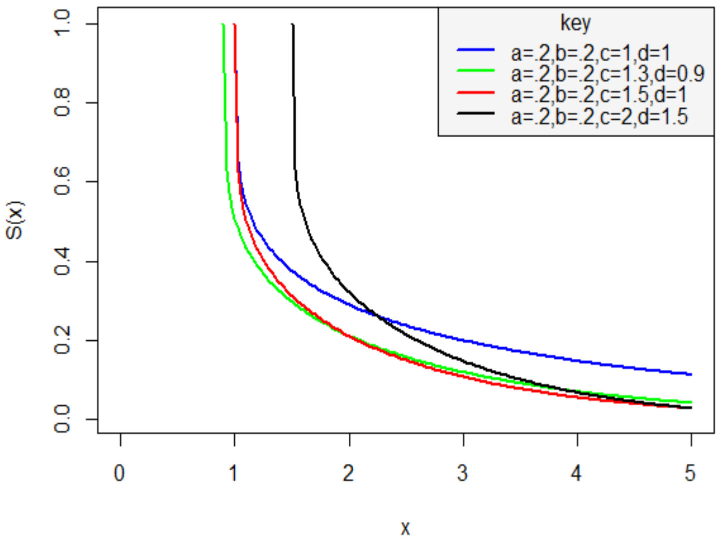

Figure 4: The Survival plot of OGE-P distribution

3.0 Statistical Properties of Four-Parameter OGE-P Distribution

3.1 *Moments*

Lemma 1: the r^{th} moment of a random variable X having OGE-P distribution is given by;

$$E(x^r) = \alpha k\theta^{-k(l+1)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\binom{\alpha-1}{i}\binom{j}{l}\frac{(-1)^{i+2j-l}(\lambda)^{j+1}(1+i)^{j+1}}{j!(k(l+1)r)}\left[\infty^{k(l+1)r} - \theta^{k(l+1)r}\right]$$

The proof:

$$\mu_r' = E(x^r) = \int_\theta^\infty x^r f(x;\alpha,\lambda,\theta,k)dx \qquad (20)$$

where $f(x;\alpha,\lambda,\theta,k)$is the PDF of the OGE-P distribution in (16).

$$E(x^r) = \int_\theta^\infty x^r \alpha k\theta^{-k}x^{k-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}(\lambda)^{1+j}(1+i)^j\binom{\alpha-1}{i}\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)^j}{j!}dx$$

$$E(x^r) = \alpha k\theta^{-k}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\alpha-1}{i}\frac{(-1)^{i+j}(\lambda)^{j+1}(1+i)^j}{j!}\int_\theta^\infty x^{k+r-1}\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)^j dx \qquad (21)$$

but,

$$\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)^j = \sum_{l=0}^{\infty}\binom{j}{l}\left(\frac{\theta}{x}\right)^{-kl}(-1)^{j-l} \qquad (22)$$

substituting (22) in to (21), we have:

$$E(X^r) = \alpha k\theta^{-k} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \binom{\alpha-1}{i}\binom{j}{l}\frac{(-1)^{i+2j-l}(\lambda)^{j+1}(1+i)^{j+1}}{j!}\int_{\theta}^{\infty} x^{k+r-1}\left(\frac{\theta}{x}\right)^{-kl} dx \quad (23)$$

further simplification of (23), yield:

$$E(x^r) = \alpha k\theta^{-k(l+1)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \binom{\alpha-1}{i}\binom{j}{l}\frac{(-1)^{i+2j-l}(\lambda)^{j+1}(1+i)^{j+1}}{j!}\int_{\theta}^{\infty} x^{k+kl+r-1} dx \quad (24)$$

$$E(x^r) = \alpha k\theta^{-k(l+1)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \binom{\alpha-1}{i}\binom{j}{l}\frac{(-1)^{i+2j-l}(\lambda)^{j+1}(1+i)^{j+1}}{j!}\left[\frac{x^{k+kl+r-1+1}}{k+kl+r-1+1}\right]_{\theta}^{\infty} \quad (25)$$

$$E(x^r) = \alpha k\theta^{-k(l+1)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \binom{\alpha-1}{i}\binom{j}{l}\frac{(-1)^{i+2j-l}(\lambda)^{j+1}(1+i)^{j+1}}{j!(k(l+1)r)}\left[x^{k(l+1)r}\right]_{\theta}^{\infty} \quad (26)$$

$$E(x^r) = \alpha k\theta^{-k(l+1)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \binom{\alpha-1}{i}\binom{j}{l}\frac{(-1)^{i+2j-l}(\lambda)^{j+1}(1+i)^{j+1}}{j!(k(l+1)r)}\left[\infty^{k(l+1)r} - \theta^{k(l+1)r}\right] \quad (27)$$

Equation (27) completes the proof and this means that the moment of this distribution tends to infinity. Hence do not exist.

3.2 *The Moment Generating Function*

Lemma 2: The moment generating function of any random variable X having OGE-P distribution is defined as;

$$E(e^{tx}) = \alpha k\theta^{-k(l+1)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \binom{\alpha-1}{i}\binom{j}{l}\frac{(-1)^{i+2j-l}(\lambda)^{j+1}(1+i)^{j+1}t^m}{j!m!(k(l+1)+m)}\left[\infty^{k(l+1)+m} - \theta^{k(l+1)+m}\right] \quad (28)$$

Proof:

$$M_x(t) = E(e^{tx}) = \int_{\theta}^{\infty} e^{tx} f(x;\alpha,\lambda,\theta,k)dx \qquad \forall x \quad (29)$$

where $f(x;\alpha,\lambda,\theta,k)$ is the PDF in (16)

$$E(e^{tx}) = \int_{\theta}^{\infty} e^{tx}\ \alpha k\theta^{-k}x^{k-1} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j}(\lambda)^{1+j}(1+i)^j\binom{\alpha-1}{i}\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)^j}{j!} dx \quad (30)$$

$$E(e^{tx}) = \alpha k\theta^{-k} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{\alpha-1}{i}\frac{(-1)^{i+j}(\lambda)^{j+1}(1+i)^j}{j!}\int_{\theta}^{\infty} e^{tx}x^{k-1}\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)^j dx \quad (31)$$

using binomial expansion, we have:

$$\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)^j = \sum_{l=0}^{\infty}\binom{j}{l}\left(\frac{\theta}{x}\right)^{-kl}(-1)^{j-l} \quad (32)$$

substituting (32) in (31), we have:

$$E(e^{tx}) = \alpha k\theta^{-k}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\binom{\alpha-1}{i}\binom{j}{l}\frac{(-1)^{i+2j-l}(\lambda)^{j+1}(1+i)^{j+1}}{j!}\int_{\theta}^{\infty} e^{tx}x^{k-1}\left(\frac{\theta}{x}\right)^{-kl}dx \quad (33)$$

using series expansion

$$e^{tx} = \sum_{m=0}^{\infty}\frac{t^m x^m}{m!} \quad (34)$$

substituting (34) in (33), we have:

$$E(e^{tx}) = \alpha k\theta^{-k(l+1)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\binom{\alpha-1}{i}\binom{j}{l}\frac{(-1)^{i+2j-l}(\lambda)^{j+1}(1+i)^{j+1}t^m}{j!\,m!}\int_{\theta}^{\infty} x^{k(l+1)+m-1}\,dx \quad (35)$$

$$E(e^{tx}) = \alpha k\theta^{-k(l+1)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\binom{\alpha-1}{i}\binom{j}{l}\frac{(-1)^{i+2j-l}(\lambda)^{j+1}(1+i)^{j+1}t^m}{j!m!}\left[\frac{x^{k(l+1)+m}}{k(l+1)+m}\right]_{\theta}^{\infty} \quad (36)$$

$$E(e^{tx}) = \alpha k\theta^{-k(l+1)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\binom{\alpha-1}{i}\binom{j}{l}\frac{(-1)^{i+2j-l}(\lambda)^{j+1}(1+i)^{j+1}t^m}{j!\,m!\,(k(l+1)+m)}\left[\infty^{k(l+1)+m} - \theta^{k(l+1)+m}\right] \quad (37)$$

Equation (37) completes the proof.

3.3 *Quantile function and Median of Four-Parameter OGE-P distribution*

The quantile function is given by;

$$x_u = \left[\frac{\theta^k}{\lambda}\left(1+\ln\left(\frac{1}{1-u^{\frac{1}{\alpha}}}\right)\right)\right]^{\frac{1}{k}} \quad (38)$$

The median of OGE-P distribution is obtained by setting $u=0.5$ in (38)

$$x_{0.5} = \left[\frac{\theta^k}{\lambda}\left(1+\ln\left(\frac{1}{1-0.5^{\frac{1}{\alpha}}}\right)\right)\right]^{\frac{1}{k}} \quad (39)$$

4.0 Estimation of Distribution Parameters

The method of Maximum Likelihood Estimation (MLEs) is employed here to estimate the parameters of the OGE-P distribution. Let $X_1, X_2, \ldots, X_n$ be n random variables from a population that follows OGE-P distribution with sample values $x_1, x_2, \ldots, x_n$ having joint probability density function as $f(x_1, x_2, \ldots, x_n; \Theta)$where $\Theta = (\alpha, \lambda, \theta, k)^T$ is a vector of an unknown parameter. Then the likelihood function $L(\Theta)$ of the random samples is defined as:

$$L(\Theta; x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n; \Theta) = L(\Theta) = \prod_{i=1}^{n} f(x_i; \Theta) \quad (40)$$

recall that

$$f(x;\alpha,\lambda,\theta,k)=\alpha\lambda k\theta^{-k}x^{k-1}e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\left(1-e^{-\lambda\left(\left(\frac{\theta}{x}\right)^{-k}-1\right)}\right)^{\alpha-1} \quad (41)$$

then the likelihood function for (4) is given by:

$$L(\Theta)=\prod_{i=1}^{n}\left[\alpha\lambda k\theta^{-k}x_i^{k-1}e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)}\left(1-e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)}\right)^{\alpha-1}\right] \quad (42)$$

taking the natural logarithm of both side of (42), we have:

$$\log L(\Theta)=\log\left[(\alpha\lambda k\theta^{-k})^n\prod_{i=1}^{n}x_i^{k-1}\,e^{\sum_{i=1}^{n}-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)}\prod_{i=1}^{n}\left(\left(1-e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)}\right)^{\alpha-1}\right)\right] \quad (43)$$

$$\log L(\Theta)=\text{n}\log(\alpha\lambda k\theta^{-k})(k-1)(\log\prod_{i=1}^{n}x_i)\left(-\lambda\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)\right)(\alpha-1)log\left(\prod_{i=1}^{n}\left(1-e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)}\right)\right) \quad (44)$$

Then the total log- likelihood function $\log L(\Theta)$ denoted by *ll*, of the random samples is defined as:

$$ll=\text{n}\log\alpha+\text{n}\log\lambda+\text{n}\log\text{k}-\text{kn}\log\theta+(k-1)\sum_{i=1}^{n}\log x_i-\lambda\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)+(\alpha-1)\sum_{i=1}^{n}\log\left(1-e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)}\right) \quad (45)$$

Hence, differentiating (45) partially with respect to each of the parameters $(\alpha,\lambda,\theta,k)$ and setting the results equal to zero gives the maximum likelihood estimates of the respective parameters. The score functions are given by:

$$\frac{\partial ll}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left(1-e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)}\right)=0 \quad (46)$$

$$\hat{\alpha}=\frac{-n}{\sum_{i=1}^{n}\log\left(1-e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)}\right)} \quad (47)$$

$$\frac{\partial ll}{\partial \alpha}=\frac{n}{\lambda}-\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)+(\alpha-1)\sum_{i=1}^{n}\left(\frac{\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)}}{1-e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)}}\right)=0 \quad (48)$$

$$\frac{\partial ll}{\partial \theta}=\frac{-kn}{\theta}+k\lambda\theta^{-(k+1)}\sum_{i=1}^{n}x_i^{k}-(\alpha-1)\sum_{i=1}^{n}\left(\frac{k\lambda\theta^{-(k+1)}x_i^{k}e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)}}{1-e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)}}\right)=0 \quad (49)$$

$$\frac{\partial ll}{\partial k}=\frac{n}{k}-nlog\theta+\sum_{i=1}^{n}logx_i^{k}+\lambda\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{-k}log\left(\frac{\theta}{x_i}\right)-(\alpha-1)\sum_{i=1}^{n}\left(\frac{\lambda\left(\frac{\theta}{x_i}\right)^{-k}log\left(\frac{\theta}{x_i}\right)e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)}}{1-e^{-\lambda\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)}}\right)=0 \quad (50)$$

However, the three preceding systems of nonlinear equations (48), (49) and (50) cannot be solved analytically, however, estimates of the MLEs can be obtained through numerical approximation using statistical software.

5.0 Conclusion

In this article, we propose a new probability distribution called the Odd Generalized Exponential-Pareto (OGE-P) distribution and we studied some of its statistical properties.

The moments, moment generating function, quantile function, median survival function, and hazard function were derived. We also provide plots of the CDF, PDF, hazard, and survival functions for different parameters value. The parameters of the distribution were also estimated using the method of maximum likelihood.

The applications of this new distribution to some real-life data sets shall be provided in future studies. It is hoped that this new distribution would provide a better fit on skewed datasets as they are often encountered in many real-life cases.

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