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Original Research Article

Exploring Some Properties of Odd Lomax-Exponential Distribution

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Abstract

In this article, we propose a new distribution from a family of Lomax-G (type III) distribution called Odd Lomax-Exponential (type III) (OLE) distribution generated from the logit of Lomax random variable. The plots of the density and hazard rate of four cases of the new distribution were constructed for ease of understanding of its shapes under different parameter combinations. Some mathematical properties of this new distribution such as the quantile function, mode, asymptotic behaviour and the density of order statistics of OLE were obtained. The estimators of the model's parameters were determined by employing the method of maximum likelihood. The importance and flexibility of the new family are illustrated by applying it to a real-life dataset.

Keywords: Lomax-G (Type III) distribution, Odd Lomax-Exponential (Type III) distribution, MLE, Quantile function.

1.0 Introduction

Probability Theory is an important concept in data mining and the study of distribution theory is also very useful in describing real-world phenomena. Although, many distributions have been developed there is always room for new distributions that are either more flexible in terms of fitting a specific real-world scenario. Based on this concept, it has motivated researchers to seek and develop new flexible distributions. This has led to the study and development of many new distributions in literature.

From the past several years, there is a growing trend of generating new families of distributions from existing distribution by adding one or more additional parameter(s) to the baseline distribution to study the behavior of the shapes of density and hazard rate, and for checking the goodness-of-fit of proposed distributions.

Let $g(x)$, $G(x)$ and $1 - G(x)$ represent the probability density function, cumulative distribution function and survival function of the baseline distribution. Eugene (2002) studied and introduced beta-normal distribution from the beta-G family using logit of beta distribution. Zografos and Balakrishnan (2009) proposed gamma-G family using generator $-\log[1-G(x)]$. Cordeiro and de-Catro (2011) proposed a very flexible generalized family by adding two-additional parameters from the logit of Kumaraswamy distribution. Torabi and Montazeri *et al.* (2012) proposed odd gamma family from the logit of gamma distribution using $G(x)/[1 - G(x)]$ generator.

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Alexander *et al.* (2012) extended the beta-G family and introduced McDonald-G family of distributions. Bourguignon *et al.* (2014) also used generator G(x)/[1−G(x)] and introduced the Weibull-G family of distribution from Weibull distribution.

Risti´c and Balakrishnan (2012) introduced another gamma-G family from generator −log[G(x)]. Amini *et al.* (2012) introduced two log-gamma-G families from generators −log[1−G(x)] and −log[G(x)] with motivation to upper and lower records. Alzaatreh *et al.* (2013) pioneered a very general approach, the transformed-transformer (T-X) family. Cordeiro *et al.* (2013) proposed an exponentiated-generalized-G family of distributions. Alzaghal. *et al.* (2013) further extended T-X family and proposed an exponentiated T-X family of distributions. Aljarrah *et al.* (2014) introduced T-X family based on the quantile function approach.

$$F(x) = \int_0^{W(G(x))} r(t)dt = R\{W(F(x))\}$$

By replacing W(F(x)) in Alzaatreh (2013) with G(x)/[1−G(x)], we shall generate a family of Odd family of distribution (OFD) with

$$F(x) = \int_0^{\mathrm{G(x)/[1-G(x)]}} r(t)dt = R\{\mathrm{G(x)/[1-G(x)]}\} \quad (1)$$

used by Bourguignon (2014) to introduce and study Weibull –G family which is different from Weibull-X.

Odd Lomax-G is derived using Alzaghal *et al.* (2013), where W(F(x)) is replace with G(x)/[1−G(x)] given that the pdf and cdf of Lomax distribution are

$$f_T(x) = \frac{\lambda k}{(1+\lambda x)^{k+1}}$$

and

$$F_T(x) = 1 - (1+\lambda x)^{-k} \quad (2)$$

$0 \le \theta \le \infty$. Hence, we have

$$F(x) = 1 - \left(1 + \lambda\left\{\frac{\mathrm{G(x)}}{[1-\mathrm{G(x)}]}\right\}\right)^{-k} \quad (3)$$

By differentiating (3) we have the corresponding pdf.

$$f(x) = \frac{k\lambda}{(1-\mathrm{G(x)})^2} g(x)\left(1 + \lambda\left\{\frac{\mathrm{G(x)}}{[1-\mathrm{G(x)}]}\right\}\right)^{-(k+1)} \quad (4)$$

which is the pdf of Odd Lomax-G family distribution

In section two, we present a new distribution using (3) and (4) called Odd Lomax Exponential (OLE) Distribution and its shapes. The statistical properties of the OLE are investigated in section three. The Maximum Likelihood Estimators (MLEs) and Monte-Carlo study on the parameters of the OLE are discussed in section four. The application of the model and its results are presented in section five while section six contains the conclusion to the work.

2.0 The OLE Distribution

In this section, we derived a new distribution called Odd Lomax Exponential Distribution (OLE) with its PDF, CDF, survival function, Hazard function and plot the shapes of OLE.

Let the pdf and CDF of parent distribution (Exponential distribution), g(x) and G(x) be given as

$$g(x) = 1/be^{\left(-\frac{x}{b}\right)}$$

and

$$G(x) = 1 - e^{-\left(\frac{x}{b}\right)} \tag{5}$$

where b is the scale parameter, $0 < b \leq \infty$.

To derive the distribution function of the new proposed distribution we substitute (5) in (4)

$$F(x) = 1 - \left(1 + \lambda\left(e^{\left(\frac{x}{b}\right)} - 1\right)\right)^{-k} \tag{6}$$

The corresponding PDF is given by differentiating (6)

$$f(x) = \frac{k\lambda}{b} e^{\left(\frac{x}{b}\right)} \left(1 + \lambda\left(e^{\left(\frac{x}{b}\right)} - 1\right)\right)^{-(k+1)} \tag{7}$$

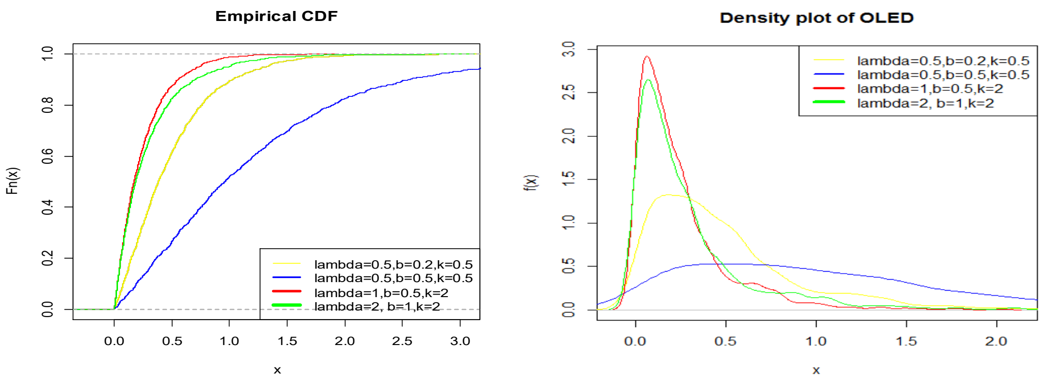

Fig.1: CDF and PDF of various OLE distributions for different values of parameters (λ, *b and k*)

2.1 *Survival Function*

The survival function of the OLE distribution is derived from this definition

$$S(x) = 1 - F_X(x)$$

where $F_X(x)$ is the CDF of OLE distribution given in (5). The survival function *S*(x) can be written as

$$S(x) = 1 - \left[1 - \left(1 + \lambda\left(e^{\left(\frac{x}{b}\right)} - 1\right)\right)^{-k}\right]$$

$$S(x) = \left(1 + \lambda\left(e^{\left(\frac{x}{b}\right)} - 1\right)\right)^{-k}$$

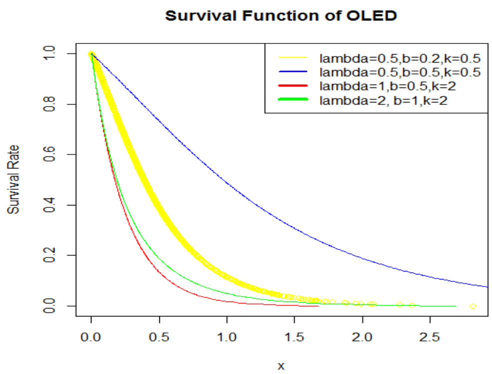

Fig. 2: Plot of Survival function of various OLE distributions for different values of parameters (λ, *b and k*).

2.2 *Hazard Function*

The hazard function of the OLE distribution is derived from this definition

$$h(x) = \frac{f_X(x)}{1 - F_X(x)}$$

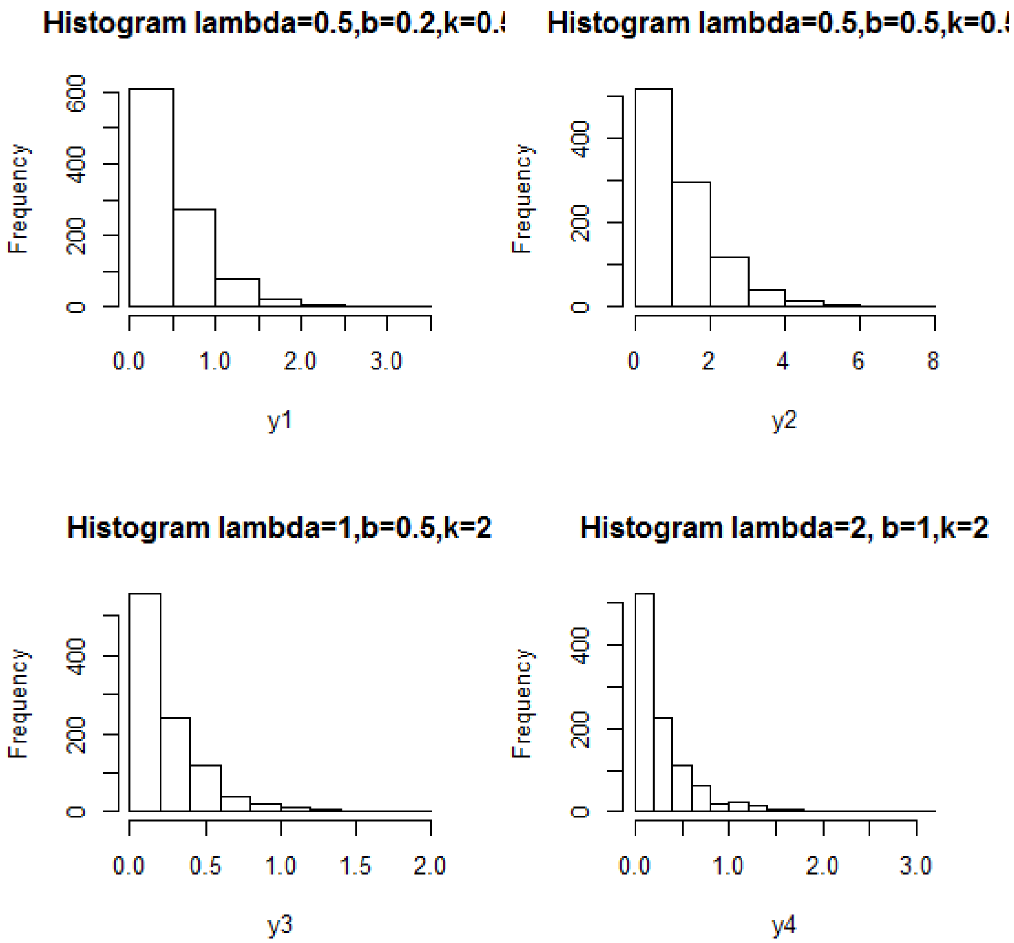

Fig.3: Histogram of OLE with different parameters (λ, *b and k.)*

where $f_X(x)$ and $F_X(x)$ are the PDF and CDF of OLE distribution given in (6) and (5) respectively. The hazard function $h(x)$ can be written as

$$h(x) = \frac{\frac{k\lambda}{b} e^{\left(\frac{x}{b}\right)} \left(1 + \lambda\left(e^{\left(\frac{x}{b}\right)} - 1\right)\right)^{-(k+1)}}{1 - \left[1 - \left(1 + \lambda\left(e^{\left(\frac{x}{b}\right)} - 1\right)\right)^{-k}\right]}$$

$$h(x) = \frac{\frac{k\lambda}{b} e^{\left(\frac{x}{b}\right)} \left(1 + \lambda \left(e^{\left(\frac{x}{b}\right)} - 1\right)\right)^{-(k+1)}}{\left(1 + \lambda \left(e^{\left(\frac{x}{b}\right)} - 1\right)\right)^{-k}}$$

$$h(x) = \frac{k\lambda}{b} e^{\left(\frac{x}{b}\right)} \left(1 + \lambda \left(e^{\left(\frac{x}{b}\right)} - 1\right)\right)^{-1} \quad (9)$$

Fig. 1 is the graphs of the Cumulative density function and Probability density function of OLE distribution respectively for different parameter values. Both graphs show different curves for various values of the parameter λ, *b and k.*

Fig. 3 displays the histogram of the data and the fitted OLE density function. Fig. 1 and Fig. 3 show that OLE distribution is a positively skewed distribution.

2.3 Cumulative Hazard Function

The cumulative hazard function, $H(x)$ of the OLE distribution is derived from this definition

$$H(\mathrm{x}) = -log_e S(x)$$

where *S(x)* is the survival function of OLE distribution given in (8). It can be written as

$$H(\mathrm{x}) = -log_e \left(1 + \lambda \left(e^{\left(\frac{x}{b}\right)} - 1\right)\right)^{-k}$$

$$H(x) = klog_e \left(1 + \lambda \left(e^{\left(\frac{x}{b}\right)} - 1\right)\right) \quad (10)$$

3.0 Statistical Properties

In this section, quantile function, median, mode order statistics and asymptotic behavior of OLE distribution are computed

3.1 *Quantile Function, Median and Mode of OLE*

The quantile function, $Q_X(p)$ of OLE distribution is the inverse function of the CDF, which was derived in (23). Recall from (5)

$$F(x) = 1 - \left(1 + \lambda \left(e^{\left(\frac{x}{b}\right)} - 1\right)\right)^{-k}$$

Let $F(x) = p$, solve for x in the above expression then, we have

$$Q_X(p) = bln\left[\frac{(1-p)^{-1/k} - 1}{\lambda} + 1\right] \quad (11)$$

Therefore, the median is given as

$$Q_X(0.5) = bln\left[\frac{(0.5)^{-1/k} - 1}{\lambda} + 1\right]$$

Proposition 1: *The OLE model is unimodal for all values of* $\lambda, b, k > 0$ *and the modal function is given as:*

$$X_{mode} = bln\left(-\frac{\lambda - 1}{k\lambda}\right) \quad (12)$$

Proof: Recall equation (7)

$$f(x) = \frac{k\lambda}{b} e^{\left(\frac{x}{b}\right)} \left(1 + \lambda \left(e^{\left(\frac{x}{b}\right)} - 1\right)\right)^{-(k+1)}$$

Differentiating (6) we have,

$$-\frac{k\lambda e^{-\frac{x}{b}}\left(1+\lambda\left(1-e^{-\frac{x}{b}}\right)\right)^{-k-1}}{b^2}+\frac{k\lambda^2\left(e^{-\frac{x}{b}}\right)^2\left(1+\lambda\left(1-e^{-\frac{x}{b}}\right)\right)^{-k-1}(-k-1)}{b^2\left(1+\lambda\left(1-e^{-\frac{x}{b}}\right)\right)}$$

and further equate it to zero, we have

$$\frac{k\lambda e^{\frac{x}{b}}\left(1+\lambda e^{\frac{x}{b}}-\lambda\right)^{-k-1}}{b^2}+\frac{k\lambda^2\left(e^{\frac{x}{b}}\right)^2\left(1+\lambda e^{\frac{x}{b}}-\lambda\right)^{-k-1}(-k-1)}{b^2\left(1+\lambda e^{\frac{x}{b}}-\lambda\right)}=0$$

and subsequently, solve for x in the above equation, we have

$$X_{mode}=bln\left(-\frac{-1+\lambda}{k\lambda}\right)$$

3.2 Order Statistics

Theorem: Let a random sample $X_1, X_2, \dots, X_n$ be from the distribution function *F(x)* and corresponding pdf *f(x)*. The PDF of i^{th} order statistic is given as

$$f_{i:n}(x_i)=a_{j,l,m,r}\left(e^{\left(\frac{x_i}{b}\right)}\right)^{r+1} \quad (13)$$

where $a_{j,l,m,r}=\frac{n!}{(i-1)!(n-i)!}\frac{k\lambda}{b}\sum_{j,l,m,r=0}^{\infty}(-1)^{2j+i+l-1}\binom{n-i}{j}\binom{j+i-1}{l}\binom{-k-kl-1}{m}\binom{-k-kl+m-1}{r}\lambda^{-(k+1)-kl+m}$

Proof:

$$f_{i:n}(x_i)=\frac{n!}{(i-1)!\,(n-i)!}f(x)F(x)^{i-1}[1-F(x)]^{n-i}$$

$$f(x)=\frac{k\lambda}{b}e^{\left(\frac{x}{b}\right)}\left(1+\lambda\left(e^{\left(\frac{x}{b}\right)}-1\right)\right)^{-(k+1)}$$

$$=\frac{n!}{(i-1)(n-i)}\frac{k\lambda}{b}\sum_{j}^{\infty}(-1)^j\binom{n-i}{j}e^{\left(\frac{x}{b}\right)}\left(1+\lambda\left(e^{\left(\frac{x}{b}\right)}-1\right)\right)^{-(k+1)}[F(x)]^{j+i-1}$$

$$=\frac{n!}{(i-1)(n-i)}\frac{k\lambda}{b}\sum_{j}^{\infty}(-1)^j\binom{n-i}{j}e^{\left(\frac{x}{b}\right)}\left(1+\lambda\left(e^{\left(\frac{x}{b}\right)}-1\right)\right)^{-(k+1)}\left[1-\left(1+\lambda\left(e^{\left(\frac{x}{b}\right)}-1\right)\right)^{-k}\right]^{j+i-1}$$

$$=\frac{n!}{(i-1)(n-i)}\frac{k\lambda}{b}\sum_{j,l}^{\infty}(-1)^{2j+i-1}\binom{n-i}{j}\binom{j+i-1}{l}e^{\left(\frac{x}{b}\right)}\left(1+\lambda\left(e^{\left(\frac{x}{b}\right)}-1\right)\right)^{-(k+1)-kl}$$

$$=\frac{n!}{(i-1)!\,(n-i)!}\frac{k\lambda}{b}\sum_{j,l,m}^{\infty}(-1)^{2j+i+l-1}\binom{n-i}{j}\binom{j+i-1}{l}\binom{-k-kl-1}{m}e^{\left(\frac{x}{b}\right)}\left(\lambda\left(e^{\left(\frac{x}{b}\right)}-1\right)\right)^{-(k+1)-kl+m}$$

$$=\frac{n!}{(i-1)!\,(n-i)!}\frac{k\lambda}{b}\sum_{j,l,m,r=0}^{\infty}(-1)^{2j+i+l-1}\binom{n-i}{j}\binom{j+i-1}{l}\binom{-k-kl-1}{m}\binom{-k-kl+m-1}{r}\lambda^{-(k+1)-kl+m}\left(e^{\left(\frac{x}{b}\right)}\right)^{r+1}$$

$$a_{j,l,m,r}=\frac{n!}{(i-1)!\,(n-i)!}\sum_{j,l,m,r=0}^{\infty}(-1)^{2j+i+l-1}\binom{n-i}{j}\binom{j+i-1}{l}\binom{-k-kl-1}{m}\binom{-k-kl+m-1}{r}$$

$$f_{i:n}(x_i)=a_{j,l,m,r}\lambda^{-(k+1)-kl+m}\frac{k\lambda}{b}\left(e^{\left(\frac{x_i}{b}\right)}\right)^{r+1}$$

3.3 *Asymptotic Behavior of OLE Distribution*

The asymptotic behavior of the proposed distribution model OLE, when $x \to 0$ and when $x \to \infty$ are given

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{k\lambda}{b} e^{\left(\frac{x}{b}\right)} \left(1+\lambda\left(e^{\left(\frac{x}{b}\right)}-1\right)\right)^{-(k+1)} = \frac{k\lambda}{b}$$

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty} \frac{k\lambda}{b} e^{\left(\frac{x}{b}\right)} \left(1+\lambda\left(e^{\left(\frac{x}{b}\right)}-1\right)\right)^{-(k+1)} = 0$$

4.0 Estimation and Simulation

This section estimates the parameters of OLE distribution and simulates the behavior of the distribution.

4.1 *Maximum Likelihood Estimation*

A random sample X is taken from the OLE distribution with probability density function (pdf) given as

$$f(x) = \frac{k\lambda}{b} e^{\left(\frac{x}{b}\right)} \left(1+\lambda\left(e^{\left(\frac{x}{b}\right)}-1\right)\right)^{-(k+1)} \quad ; x > 0; \lambda > 0; b > 0; k > 0$$

By definition, the likelihood function of OLE is given by:

$$f(x;\lambda,b,k) = \left(\frac{k\lambda}{b}\right)^n \left[e^{\frac{\sum_{i=1}^{n}(x_i)}{b}}\right] \prod_{i=1}^{n} \left[1+\lambda\left(e^{\left(\frac{x}{b}\right)}-1\right)\right]^{-k-1} \tag{14}$$

and the log-likelihood function is given by:

$$\mathrm{L} = \mathrm{Lnf}(x;\lambda,b,k) = nlnk + nln\lambda - nlnb + \frac{1}{b}\sum_{i=1}^{n} x_i - (k+1)\sum_{i=1}^{n} \ln\left(1+\lambda\left(e^{\left(\frac{x}{b}\right)}-1\right)\right) \tag{15}$$

Taking a partial differentiation of equation (15) with respect to λ , b and k respectively and equate them zero

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} - (k+1)\left(\sum_{i=1}^{n} \frac{e^{\frac{x_i}{b}} - 1}{1+\lambda\left(e^{\frac{x_i}{b}}-1\right)}\right) = 0 \tag{16}$$

$$\frac{\partial L}{\partial b} = \lambda (k+1)\left(\sum_{i=1}^{n} \frac{x_i e^{\frac{x_i}{b}}}{\lambda e^{\frac{x_i}{b}} - \lambda + 1}\right) - nb + \sum_{i=1}^{n} x_i = 0 \tag{17}$$

$$\frac{\partial L}{\partial k} = \frac{n}{k} - \sum_{i=1}^{n} \ln\left(1+\lambda\left(e^{\frac{x_i}{b}}-1\right)\right) = 0 \tag{18}$$

The solution of the non-linear system of equations obtained by differentiating (16), (17) and (18) with respect to λ, b and k gives the maximum likelihood estimates of the model parameters. The solution can also be obtained directly by using R software when data sets are available.

4.2 *Simulation and Result*

A simulation study was conducted to evaluate the MLE estimates, bias and standard error for various parameter combinations and different sample sizes. We consider the values 0.5, 1.5, 2 , 2.5 for the parameter λ, 0.2. 1, 1.5, 2 for the parameter b and 0.5, for the parameter k being scale parameter. The process is repeated 1000 times. Four different sample sizes n = 100, 200, 500 and 1000 are considered. The estimates, bias and standard error are presented in Table 1.

Table 1: Maximum likelihood estimates, bias and standard error of difference for combinations of values of parameters (*λ, b, k*)

	n=100			200			500			1000		
	Estimate	Bias	Std. Error	Estimate	Bias	Std. Error	Estimate	Bias	Std. Error	Estimate	Bias	Std. Error
0.5	0.401	0.099	0.201	0.218	0.282	0.099	0.329	0.171	0.094	0.489	0.011	0.056
0.2	0.117	0.083	0.157	0.067	0.133	0.032	0.062	0.138	0.025	0.193	0.007	0.068
0.5	0.292	0.208	0.436	0.166	0.334	0.088	0.143	0.357	0.062	0.491	0.009	0.197
1.5	1.336	0.164	0.448	1.736	0.236	0.368	1.687	0.187	0.291	1.471	0.029	0.156
1.0	1.121	0.121	4.528	0.014	0.986	NA	2.695	1.695	1.407	1.053	0.053	0.645
0.5	0.594	0.094	2.230	0.008	0.492	NA	1.161	0.661	0.502	0.535	0.035	0.301
2.0	1.781	0.219	0.591	1.640	0.360	0.442	2.214	0.214	0.330	1.962	0.038	0.208
1.5	1.570	0.070	1.966	3.150	1.650	3.365	3.116	1.616	1.151	1.546	0.046	0.568
0.5	0.555	0.055	0.607	1.052	0.552	0.943	0.892	0.392	0.270	0.524	0.024	0.166
2.5	2.225	0.275	0.757	2.047	0.453	0.525	2.756	0.256	0.417	2.454	0.046	0.258
2.0	2.028	0.028	2.616	3.673	1.673	2.128	3.665	1.665	1.238	2.044	0.044	0.524
0.5	0.540	0.040	0.572	0.909	0.409	0.420	0.787	0.287	0.205	0.519	0.019	0.112

Table 1 above does not contain all combinations of *λ, b and k*. The Table1 shows that when n ≥ 100 the biases are a bit high but as the value of n increases the biases converge to zero

5 Application

In this section, we shall compare OLE distribution with two other existing distributions (three-parameter Weibull and Weibull-Rayleigh distributions). The maximum likelihood method is applied to estimate the parameters. The log-likelihood value, the Akaike Information Criterion (AIC) are reported. The plot of fitted density.

The Data: Number of Patients Arrivals per hour at the University of Lagos Teaching Hospital, Lagos. Nigeria.

Data set on the daily arrival of patients per hour at the University of Lagos Teaching Hospital for April 2017 (30days) is used to investigate the flexibility of the new distribution proposed in this work.

Daily Arrival of Patients per hour at the ULTH, Lagos, Nigeria

62	25	65	60	53	27	50	40	41	51
39	71	57	40	50	72	76	140	62	62
69	99	33	38	89	40	54	56	84	56

Table 2: Parameter Estimates for Hospital Arrival data

Distribution	OLE	Lomax	W-R
Parameter estimates	$\hat{\lambda}$ = 0.006799 $\hat{b}$ = 0.3192 $\hat{k}$ = 1.3774	$\hat{a}$ = 4.1558 $\hat{b}$ = 0.4680	$\hat{a}$ = 0.2785 $\hat{k}$ = 209.4921
Log-likelihood	-48.8389	-309.7594	-299.8571
AIC	103.6778	623.5188	603.7141
BIC	107.8814	624.3212	604.5165
HQIC	105.0226	622.4153	602.6106
	Skewness = 1.444,	Kurtosis = 6.115	

From Table 2, it can be observed that the proposed model OLE has lower value of AIC as compared to other models. Thus our model provides a better fit, for this particular dataset, when compared to other models to this hospital patients' arrival time data set.

6.0 Concluding Remarks

In this paper, a new probability distribution family called Odd Lomax Exponential (OLE) distribution is proposed Some characteristics of the new OLE such as expressions for the density function, cumulative function, moments, quantile function, mode, asymptotic behavior and order statistics are discussed. The method of MLE was adopted to estimate the parameters of the new distribution. The application of the new OLE distribution on real-life data set revealed the goodness of the new distribution.

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