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Original Research Article

Development of GARCH(p,q) Model With Generalized *t*-Distributed Error Innovation with Application

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Abstract

Generalized Auto-Regressive Conditional Heteroscedasticity (GARCH(p,q)) models are known for modelling volatility in returns of financial assets. Over the years, researchers have often modelled volatility with normal error innovation. Meanwhile, most financial series are non-normal and exhibit fat tails as well as highly leptokurtic. Thus, the use of normal error innovation for modelling data that exhibit high volatility always yields inaccurate and poor forecast performance. Therefore, this study seeks to develop appropriate theoretical alternative error innovation for GARCH (1,1) and Taylor Schwartz GARCH (TS-GARCH (1,1)) models that are expected to ameliorate the deficiencies in the use of normal error innovation.

Keywords Volatility, GARCH, TS-GARCH, Error Distribution.

1.0 Introduction

Suppose the return on investment received at some pre-defined point in the future of one's investment in a financial asset today is considered as a random variable, such a variable can be fully described by a distribution function or by a density function. The expected or mean value of a density function is the most important feature of the density function which represents the location of the density function. The uncertainty or the volatility occurs around the mean, this is observed when plotting returns against time, the volatility is illustrated by the jagged oscillating appearance. This phenomenon of jagged oscillating appearance is usually observed over stated periods of time, these may be hourly, daily, or weekly, say. After the volatility of a time series is observed it would be obvious, interesting and expedient to examine the properties of the series; can it be forecasted from its own past, do other series improve these forecasts, can the series be modelled conveniently and can the results be generalized using some useful multivariate methods?. Financial econometricians have done a lot of work in this area and considered such questions. There is now a substantial and often sophisticated literature in this area (Poon, 2005).

Volatility generally, is the rapid movement at which the value of a security increases or decreases over time. Financial markets experts are often concerned with the spread of return on asset since volatility is often measured by the standard deviation. The success or failure of volatility models in many financial applications depicts the practical importance of volatility modelling and forecasting. Therefore, the success or failure of any volatility models does depend on the characteristics of empirical data the researcher tries to capture and predict.

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Therefore, the crux of volatility modelling is to explore the properties of a time series and obtain the stylized facts of financial market volatility. Financial time series have their distinct characteristics, these salient features about financial market returns and volatility include fat tails, volatility clustering, the autocorrelation of squared returns, no autocorrelation for asset returns except possibly at lag one. Thus, the specification of an appropriate volatility model for capturing these features exhibited by financial series is of significance to policy and economic managers. More so, reliable volatility model of asset returns assists investors in their risk management decisions and portfolio adjustments.

In modelling financial returns, there are both theoretical and empirical reasons for preferring logarithmic returns. According to Strong (1992), theoretically, logarithmic returns are analytically more tractable when linking together sub-period returns to form returns over long intervals. Empirically, logarithmic returns have much better statistical properties (Christoffersen, 2012). Also, in order to solve the problem of non-stationarity that is usually encountered with the level series, the return series is preferred to level series (Escanciano and Lobato, 2009). Therefore, the return at time t is computed on a continuously compounded basis for a particular time t as expressed below:

$$r_t = In\left(\frac{y_t}{y_{t-1}}\right) \times 100 \tag{1}$$

where; r_t= Return rate in period t
y_t= Price in period t
y_{t-1}= One period lag in the Price

2.0 Model Specification and Tests

2.1 *Mean Equation*

To correctly model the conditional variance or volatility, one must model the conditional mean, the conditional mean is often specified. If Ω_{t-1} is the information set at time t-1, which may include past returns and past residuals and any other variable known at time t-1, then, r_t, is usually modeled as follows:

$$r_t = E(r_t|\Omega_{t-1}) + \varepsilon_t \tag{2}$$

Where E (.|.) represents the conditional expectation operator and ε_t denotes the disturbance term, with $E(\varepsilon_t)$ = 0 and $E(\varepsilon_t\varepsilon_s)$ = 0, for all t not equal to s and E $(\varepsilon_t\varepsilon_s)$ =σ^2 for all t=s.
Researchers have often opined and modeled the conditional mean $E(r_t|\Omega_{t-1})$ with Autoregressive (AR), Moving Average (MA) or Autoregressive Moving Average (ARMA) terms.

2.2 *Test for unit root and Heteroscedasticity*

The augmented Dickey-Fuller test is used to test for unit root. It is specified as

$$\Delta r_t = \emptyset r_t + \sum_{j=1}^{p} \beta_i \Delta r_{t-j} + \varepsilon_t \tag{3}$$

where; $\emptyset$= coefficient presenting process root (focus of the test).
The null and alternative hypothesis corresponds to

$H_o : \emptyset = 1$ (series is non-stationary), against $H_1 : \emptyset \neq 1$.

The TR^2 test statistics where R^2 is the coefficient of variation, which is the square of correlation, and T is the number of observations will be used to test for heteroscedasticity in the return series. It is computed from the regression of squared-error (residuals) ε_t^2 on a constant and lag(s) of squared-error (residuals) expressed as;

$$\varepsilon_t^2 = \alpha_0 + \alpha_1\varepsilon_{t-1}^2 + \cdots + \alpha_q\varepsilon_{t-q}^2 \tag{4}$$

where α_1 to α_q, are the coefficients. When the coefficients are different from zero then the null of conditional Homoscedasticity is rejected. The Engle's LM test Statistics is evaluated under the null hypothesis of homoscedasticity and it asymptotically follows the chi-square distribution X^2(q).

3.0 Volatility Model

There are different models both symmetric and asymmetric that have been employed to describe the variability in asset returns. The asymmetric is adopted to measure the effect of both negative and positive shocks on conditional variance. In this study, some of the symmetric and asymmetric models employed are:

3.1 *The Generalized Autoregressive Conditional Heteroscedastic (GARCH) Model*

The generalized autoregressive conditional heteroscedastic (GARCH) model proposed by Bollerslev (1986) is employed in this study to probe and/or explore the volatility clustering and persistence usually exhibited by financial series. The GARCH model has basically three parameters, these three parameters allow for a limitless number of squared errors to influence the current conditional variance (volatility). The conditional variance which is a weighted average of past squared residuals is determined through GARCH model. These weights decline gradually but they never reach zero. Moreover, the conditional variance allowed by the GARCH model is dependent upon its own previous lags. The GARCH (p, q) model has a general framework which is expressed by allowing the current conditional variance to depend on the first p past conditional variances and the q past squared innovations. This is expressed in the form:

$$\sigma_t^2 = \omega + \sum_{j=1}^{q} \alpha_j \, \varepsilon_{t-j}^2 + \sum_{i=1}^{p} \beta_i \, \sigma_{t-i}^2 \tag{5}$$

where, p = number of lagged conditional variance, q = number of lagged squared residuals
In this study, the GARCH(1,1) model or specification is employed.

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2 \tag{6}$$

where, $\omega > 0$, $\alpha > 0$, and $\beta > 0$. Also,

$$\varepsilon_t = Z_t\sigma_t \tag{7}$$

and Z_t denotes the standardized residual returns series that is identically and independently distributed with zero mean and variance 1), and σ_t^2 is conditional variance(current conditional variance). The persistence of $\sigma_t^2$is captured by $\alpha + \beta$, while stationarity is ensured by $\alpha + \beta < 1$. There are three terms and/or functions that usually characterizes the conditional variance equation: (i) A constant term, ω (ii) the function that represents news about volatility from the previous period, (the ARCH term); and (iii) immediate past period forecast of variance σ_{t-1}^2 (the GARCH term). When residuals are generated from the mean equation, the conditional variance equation, therefore, models the time-varying nature of volatility of the residuals that have hence been generated from the mean equation.

3.2 *TS-GARCH Model*

The TS-GARCH is used to model fat tails. It was developed by Taylor (1986) and Schwert (1990). It is also another popular model used to gain the information content in the thick tails. The thickness of a tail of a series is common in the return distribution of speculative prices. The general framework of this model is based on standard deviations and expressed as:

$$\sigma_t = \omega + \sum_{j=1}^{q} \alpha_j |\varepsilon_{t-j}| + \sum_{i=1}^{p} \beta_i \sigma_{t-1} \tag{8}$$

The study makes use of the TS-GARCH(1,1) model, with the respect that TS-GARCH(1,1) provides better estimates with the following specifications below than other variations of its model.

$$\sigma_t = \omega + \alpha|\varepsilon_{t-1}| + \beta\sigma_{t-1} \tag{9}$$

3.3 *Estimation Method*

There are different methods used to estimate GARCH model parameters with respect to the distribution of the residual term. The traditional method for parameter estimation is the Least Square (LS) method, other methods are the M-estimators obtained from the likelihood equations for the location and the scale parameters and the Modified Maximum Likelihood (MML) method which has closed-form algebraic expressions and is asymptotically equivalent to the ML (maximum likelihood) estimators which are often used in parameter estimation. In this study, we employ the MML method of parameter estimation constructed under the generalized t-distribution (GTD) of the residual term.

3.4 *Modified Maximum Likelihood (MML) method*

Tiku et al (1999) obtained the parameter estimates in autoregressive models assuming the underlying distribution to be the shift-scaled Student's t distribution. They also developed the modified maximum likelihood (MML) estimators of the parameters and showed that their proposed estimators had closed forms solutions and established that they were remarkably efficient and robust. The method is now known to give estimators that are asymptotically fully efficient (Bhattacharyya 1985) and almost fully efficient for small sample sizes (Lee et al 1980, Tan 1985, Tiku and Suresh 1992, Vaughan 1992).

To formulate modified likelihood equations, Z_t (for a given $\emptyset$) is ordered in increasing order of magnitude and denote the ordered z-values by $Z_{[i]} = (r_{[i]} - \emptyset r_{[i]-1})/\sigma$, $1 \le i \le n$. It may be noted that $(r_{[i]}, r_{[i]-1})$ is that pair of (r_i, r_{i-1}) observations which constitute $Z_{[i]}$, $1 \le i \le n$. The pair $(r_{[i]}, r_{[i]-1})$ may be called concomitants of $Z_{[i]}$.

Let $t_{[i]} = E\{Z_{[i]}\}(1 \le i \le n)$ be the expected values of the standardized order statistics, noting that under very general regularity conditions $t_{[i]}$ converges to $Z_{[i]}$as n tends to infinity, and with the understanding that the fact that the function g(z) is almost linear in a small interval $c \le z \le d$ (Tiku 1967, 1968b; Tiku and Suresh 1992), we use the Taylor series expansion to linearize g(z), therefore;

$$g\left(Z_{[i]}\right) \cong m_i + n_i Z_{[i]} \tag{10}$$

where,

$$m_i = \frac{|t|^{p-1}}{\left\{1+\frac{1}{q}|t|^p\right\}} - t * \frac{(p-1)|t|^{p-2}\left\{1+\frac{1}{q}|t|^p\right\} - \frac{p}{q}|t|^{2p-2}}{\left\{1+\frac{1}{q}|t|^p\right\}} \tag{11}$$

and

$$n_i = \frac{(p-1)|t|^{p-2}\left\{1+\frac{1}{q}|t|^p\right\} - \frac{p}{q}|t|^{2p-2}}{\left\{1+\frac{1}{q}|t|^p\right\}} \tag{12}$$

To obtain the value of m_i and n_i, we need the value of $t_{[i]}$,$1 \le i \le n$.The approximate values of $t_{[i]}$ are the solutions of the following expression below;

$$\int_{-\infty}^{t_{[i]}} f(Z)dz = \frac{1}{n+1}, 1 \le i \le n \tag{13}$$

Then $t_{[i]}$ values are obtained as;

$$t_{[i]} \begin{cases} \left(\frac{y_1}{1-y_1}\right)^{1/p} \sigma q^{\frac{1}{p}}, y_1 = 1 - \left\{1 + \left(\left(\frac{i}{n+1}\right)/\sigma q^{1/p}\right)^p\right\}^{-1}, for\ r_t \ge 0 \\ -\left(\frac{y_2}{1-y_2}\right)^{1/p} \sigma q^{\frac{1}{p}}, y_2 = 1 - \left\{1 + \left(-\left(\frac{i}{n+1}\right)/\sigma q^{1/p}\right)^p\right\}^{-1}, for\ r_t < 0 \end{cases} \tag{14}$$

Hence, the MML method will be used in this research.

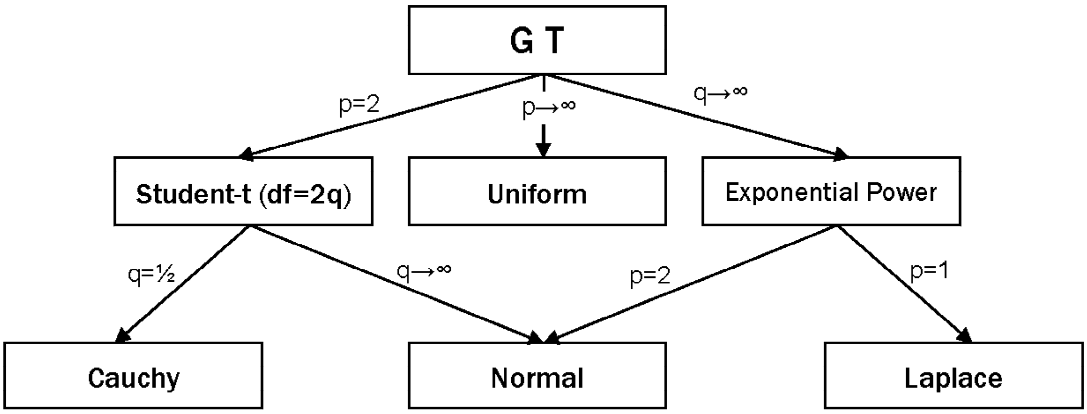
3.5 *The Proposed Error innovation*

The Generalized t - distribution (GTD) is expressed as:

$$f(r:\mu,\sigma,p,q)=\frac{p}{2\sigma q^{\frac{1}{2}}B(\frac{1}{p},q)\left[1+\frac{|r-\mu|^{p}}{q\sigma^{p}}\right]^{q+1/q}} \qquad -\infty<\varepsilon_t<\infty \tag{15}$$

where; P>0, q>0, σ>0 and B(.) is a beta function. The GTD is a symmetric function (mean, μ =0), the choice of this is to describe the purpose of risk modelling, to determine its behaviour and to give a reasonable forecast of future realization. Also, the symmetric probability distribution is the best guess for an uncertain future and if a non-symmetric distribution was assumed, then a strong hypothesis for the uncertain future concerning the movement of the returns rate is made (Saïda, 2000).
The GTD has two parameters both of which are shape parameters, thereby providing flexibility in the tail as well as in its peak. The GTD nests seven other well-known distributions, including the Student's *t*-distribution (when p=2, then df becomes 2q) and the GED (when q= ∞ and df = p), when both conditions are met (i.e p=2 and q= ∞) then GTD becomes normal distribution.

The flow-chart below gives the various distribution nested by the GTD when the shape parameters(p and q) assume different values.



The GT distribution family tree

3.6 *Estimation of The Parameters of GTD*

The log-likelihood function of the GTD in (15) to be maximized is given by the equation below;

$$Inl(.)=TIn\left(\frac{p}{2}\right)-\frac{T}{p}Inq-TIn\Gamma\left(\frac{1}{p}\right)-TIn\Gamma(q)+TIn\Gamma\left(q+\frac{1}{p}\right)-\frac{T}{2}In\sigma_t^2-\left(q+\frac{1}{p}\right)\sum_{i=1}^{T}In\left[1+\frac{|\varepsilon_t|^p}{q(\sigma_t^2)^{p/2}}\right] \tag{16}$$

From (16), let $K=TIn\left(\frac{p}{2}\right)-\frac{T}{p}Inq-TIn\Gamma\left(\frac{1}{p}\right)-TIn\Gamma(q)+TIn\Gamma\left(q+\frac{1}{p}\right)$, then,

$$Inl(.)=K-\frac{T}{2}In\sigma_t^2-(q+\frac{1}{p})\sum_{i=1}^{T}In\left[1+\frac{|\varepsilon_t|^p}{q(\sigma_t^2)^{p/2}}\right] \tag{17}$$

Therefore, partially differentiating equation (17) above with respect to the mean (μ or $\hat{r}_t$), we obtain the following expressions;

From the RHS of equation (17), let $B = 1 + \frac{1}{q}\left|\frac{r_t - \hat{r}_t}{(\sigma_t^2)^{1/2}}\right|^p$ and $Z = \frac{r_t - \hat{r}_t}{\sigma}$. Therefore, partially differentiating equation (17) with respect to the mean $\hat{r}_t$, we have

$$\frac{dl(.)}{d\hat{r}_t} = \frac{dl(.)}{dB} \times \frac{dB}{dZ} \times \frac{dZ}{d\hat{r}_t}$$

From the RHS of the expression above, we differentiate each expression separately, and obtain their product:

Firstly, $\frac{dl(.)}{dB} = \frac{d}{dB}\left\{K - \frac{T}{2} In\sigma_t^2 - (q + \frac{1}{p})\sum_{i=1}^{T} InB\right\}$. Thus,

$$\frac{dl(.)}{dB} = -\left(q + \frac{1}{p}\right)\sum_{i=1}^{T}\frac{1}{B}$$

Secondly, $\frac{dB}{dZ} = \frac{d}{dZ}\left\{1 + \frac{1}{q}|Z|^p\right\}$

Recall that the function $|Z| = sgn(Z).Z$ and $\frac{d|Z|}{dZ} = sgn(Z)$ where $sgn(Z) = \begin{cases} -1, If\ Z < 1 \\ 0, If\ Z = 0 \\ 1, If\ Z > 1 \end{cases}$

Therefore,

$$\frac{dB}{dZ} = \frac{p}{q} sgn(Z)|Z|^{p-1}$$

Lastly, $\frac{dZ}{d\hat{r}_t} = \frac{d}{d\hat{r}_t}\left\{\frac{r_t - \hat{r}_t}{\sigma}\right\}$. Hence,

$$\frac{dZ}{d\hat{r}_t} = -\frac{1}{\sigma_t}$$

The resulting product of the partial differentiation with respect to the mean is given as,

$$\frac{dl(.)}{d\hat{r}_t} = -(q + \frac{1}{p})\sum_{i=1}^{T}\frac{1}{B} \times \frac{p}{q} sgn(Z)|Z|^{p-1} \times -\frac{1}{\sigma_t}$$

$$\frac{dl(.)}{d\hat{r}_t} = \frac{(pq+1)}{q(\sigma_t^2)^{1/2}}\sum_{i=1}^{T}\frac{sgn(Z)|Z|^{p-1}}{\left\{1 + \frac{1}{q}|Z|^p\right\}} \quad (18)$$

To maximize the log-likelihood function in equation (15), we see that $\frac{dl(.)}{d\hat{r}_t} = 0$. Also, let $g(Z) = \frac{|Z|^{p-1}}{\left\{1 + \frac{1}{q}|Z|^p\right\}}$, then, equation (18) is expressed in the equation below:

$$\frac{dl(.)}{d\hat{r}_t} = \frac{(pq+1)}{q(\sigma_t^2)^{1/2}}\sum_{i=1}^{T} sgn(Z)g(Z) = 0 \quad (19)$$

Also, partially differentiating eq(17) with respect to the variance (σ_t^2), we obtain the following expression below:

$$\frac{dl(.)}{d\sigma_t^2} = -\frac{T}{2\sigma_t^2} + \left\{\frac{dl(.)}{dB} \times \frac{dB}{dZ} \times \frac{dZ}{d\sigma_t^2}\right\}$$

From the RHS of the expression above, we differentiate each expression separately, and then carry out their product

Firstly, $\frac{dl(.)}{dB} = \frac{d}{dB}\left\{K - \frac{T}{2}In\sigma_t^2 - (q + \frac{1}{p})\sum_{i=1}^{T} InB\right\}$

$$\frac{dl(.)}{dB} = -\left(q + \frac{1}{p}\right)\sum_{i=1}^{T}\frac{1}{B}$$

Secondly, $\frac{dB}{dZ} = \frac{d}{dZ}\left\{1 + \frac{1}{q}|Z|^p\right\}$

$$\frac{dB}{dZ} = \frac{p}{q}sgn(Z)|Z|^{p-1}$$

Lastly, $\frac{dZ}{d\sigma_t^2} = \frac{d}{d\sigma_t^2}\left\{\frac{r_t - \hat{r}_t}{(\sigma_t^2)^{1/2}}\right\}$

$$\frac{dZ}{d\sigma_t^2} = -\frac{(r_t - \hat{r}_t)}{2(\sigma_t^2)^{3/2}}$$

The resulting product of the partial differentiation with respect to the variance is expressed in equation (20) below,

$$\frac{dl(.)}{d\sigma_t^2} = -\frac{T}{2\sigma_t^2} + \left\{-(q + \frac{1}{p})\sum_{i=1}^{T}\frac{1}{B} \times \frac{p}{q}sgn(Z)|Z|^{p-1} \times -\frac{(r_t - \hat{r}_t)}{2(\sigma_t^2)^{3/2}}\right\}$$

$$\frac{dl(.)}{d\sigma_t^2} = -\frac{T}{2\sigma_t^2} + \frac{(pq+1)}{2q\sigma_t^2}\sum_{i=1}^{T}\frac{sgn(Z)|Z|^{p-1}}{\left\{1+\frac{1}{q}|Z|^p\right\}} \cdot \frac{(r_t - \hat{r}_t)}{(\sigma_t^2)^{1/2}} \quad (20)$$

Therefore, by maximization of the log-likelihood function in equation (20), we set $\frac{dl(.)}{d\sigma_t^2}$, hence resulting to equation (21) below.

$$\frac{dl(.)}{d\sigma_t^2} = -\frac{T}{2\sigma_t^2} + \frac{(pq+1)}{2q\sigma_t^2}\sum_{i=1}^{T} Zg(Z) = 0 \quad (21)$$

3.7 *Estimating Volatility Models*

From earlier derivation, of the ML function of the generalized t-distribution with respect to the variance, we obtained equation (21) as shown below;

$$\frac{dl(.)}{d\sigma_t^2} = -\frac{T}{2\sigma_t^2} + \frac{(pq+1)}{2q\sigma_t^2}\sum_{i=1}^{T} Zg(Z) = 0$$

Also, recall that $g(Z_{[i]}) \cong m_i + n_i Z_{[i]}$

Therefore, substituting the function above in equation (21), we obtain the following expressions,

$$\frac{dl(.)}{d\sigma_t^2} = -\frac{T}{2\sigma_t^2} + \frac{(pq+1)}{2q\sigma_t^2}\sum_{i=1}^{T} Z_{[i]}\{m_i + n_i Z_{[i]}\}$$

$$\frac{dl(.)}{d\sigma_t^2} = -\frac{T}{2\sigma_t^2} + \frac{(pq+1)}{2q\sigma_t^2}\sum_{i=1}^{T}\{m_i Z_{[i]} + n_i Z_{[i]}^2\}$$

$$\frac{dl(.)}{d\sigma_t^2} = -T + \frac{(pq+1)}{q}\sum_{i=1}^{T} m_i Z_{[i]} + \frac{(pq+1)}{q}\sum_{i=1}^{T} n_i Z_{[i]}^2$$

Recall that Z in the above expressions is, $Z = \frac{r_t - \hat{r}_t}{\sigma} = \frac{\varepsilon_t}{\sigma}$

Where, $\varepsilon_t = r_t - \emptyset\hat{r}_{t-1}$ for an AR(1) model, $\varepsilon_t = r_t - \theta\varepsilon_{t-1}$ for a MA(1) model and $\varepsilon_t = r_t - \emptyset\hat{r}_{t-1} - \theta\varepsilon_{t-1}$ for an ARMA(1,1). Therefore, the expression for the variance of the GTD is as below;

$$\frac{dl(.)}{d\sigma_t^2} = -T + \frac{(pq+1)}{q}\sum_{i=1}^{T} m_i \frac{\varepsilon_t}{\sigma_t} + \frac{(pq+1)}{q}\sum_{i=1}^{T} n_i \frac{\varepsilon_t^2}{\sigma_t^2}$$

$$\frac{dl(.)}{d\sigma_t^2} = -T\sigma_t^2 + \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} m_i\varepsilon_t\right]\sigma_t + \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} n_i\varepsilon_t^2\right]$$

Therefore, the expression above becomes equation (22) below by equating the expression to zero,

$$T\sigma_t^2 - \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} m_i\varepsilon_t\right]\sigma_t - \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} n_i\varepsilon_t^2\right] = 0 \tag{22}$$

Hence, from the quadratic equation, $ax^2 + bx + c = 0$ and $x = \frac{-b\pm\sqrt{(b^2-4ac)}}{2a}$

From equation (22),

Let $D = \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} m_i\varepsilon_t\right]$ and $H = \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} n_i\varepsilon_t^2\right]$ (23)

Then equation (22) becomes,

$$T\sigma_t^2 - D\sigma_t - H = 0 \tag{24}$$

Therefore, going by the use of the quadratic equation formula, we have equation (25) below,

$$\sigma_t = \frac{D+\sqrt{(D^2-4TH)}}{2T} \tag{25}$$

Therefore, to correct the issue of biasness, we rewrite the denominator in equation (25) and replace with $2\sqrt{T(T-1)}$. Therefore, equation (25) becomes,

$$\sigma_t = \frac{D+\sqrt{(D^2-4TH)}}{2\sqrt{T(T-1)}} \tag{26}$$

3.8 *Estimating The Parameters of GARCH (1,1) Model*

The GARCH (1,1) process is expressed below as;

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$

Therefore, estimates of their respective parameters are obtained by substituting the GARCH (1,1) model into the variance of the generalized t distribution.

Firstly, estimating the omega parameter, we further differentiate equation (20) with respect to omega, thus the expression below describes the procedure;

$$\frac{dl(.)}{d\omega} = \frac{dl(.)}{d\sigma_t^2} \times \frac{d\sigma_t^2}{d\omega}$$

Hence, by differentiating each term from the RHS of the above expression, we obtain the following,

$$\frac{dl(.)}{d\omega} = \left\{-\frac{T}{2\sigma_t^2} + \frac{(pq+1)}{2q\sigma_t^2}\sum_{i=1}^{T} Zg(Z)\right\} \times 1$$

Therefore, by equating the expression above to zero, we arrive at equation (24) as stated below,

$$T\sigma_t^2 - D\sigma_t - H = 0$$

where T is the number of observations, D and H are as stated in equation (22). Therefore by substituting the GARCH(1,1) model in the quadratic equation (24) above, we have the expression below,

$$T(\omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2) - D\sigma_t - H = 0$$

Expanding the expression above and making omega (ω) the subject of the formula yields the following

$$\omega = \frac{D\sigma_t + H - \alpha\sum_{i=1}^{T}\varepsilon_{t-1}^2 - \beta\sum_{i=1}^{T}\sigma_{t-1}^2}{T} \tag{27}$$

Estimating the parameter α, we differentiate equation (20) with respect to α. Thus, the expression below describes the procedure;

$$\frac{dl(.)}{d\alpha} = \frac{dl(.)}{d\sigma_t^2} \times \frac{d\sigma_t^2}{d\alpha}$$

Hence, by differentiating each term from the RHS of the above expression, we obtain the following expressions,

$$\frac{dl(.)}{d\alpha} = \left\{-\frac{T}{2\sigma_t^2} + \frac{(pq+1)}{2q\sigma_t^2}\sum_{i=1}^{T} Zg(Z)\right\} \times \varepsilon_{t-1}^2$$

Therefore, by expanding and equating the expression above to zero, we obtain the following,

$$\frac{dl(.)}{d\alpha} = -\frac{T\varepsilon_{t-1}^2}{2\sigma_t^2} + \frac{(pq+1)}{2q\sigma_t^2}\sum_{i=1}^{T}\varepsilon_{t-1}^2\{m_i Z_{[i]} + n_i Z_{[i]}^2\}$$

$$\frac{dl(.)}{d\alpha} = -T\varepsilon_{t-1}^2 + \frac{(pq+1)}{q}\sum_{i=1}^{T} m_i \frac{\varepsilon_t\varepsilon_{t-1}^2}{\sigma_t} + \frac{(pq+1)}{q}\sum_{i=1}^{T} n_i \frac{\varepsilon_t^2\varepsilon_{t-1}^2}{\sigma_t^2}$$

$$\frac{dl(.)}{d\alpha} = -\sigma_t^2 T\varepsilon_{t-1}^2 + \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} m_i\varepsilon_t\varepsilon_{t-1}^2\right]\sigma_t + \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} n_i\varepsilon_t^2\varepsilon_{t-1}^2\right]$$

$$\sigma_t^2\sum_{i=1}^{T}\varepsilon_{t-1}^2 - \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} m_i\varepsilon_t\varepsilon_{t-1}^2\right]\sigma_t - \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} n_i\varepsilon_t^2\varepsilon_{t-1}^2\right] = 0 \tag{28}$$

From equation (28)

Let, $S = \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} m_i\varepsilon_t\varepsilon_{t-1}^2\right]$ and $V = \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} n_i\varepsilon_t^2\varepsilon_{t-1}^2\right]$ (29)

Then equation (29) becomes,

$$\sigma_t^2\sum_{i=1}^{T}\varepsilon_{t-1}^2 - S\sigma_t - V = 0 \tag{30}$$

where S and V are as stated in equation (29).

Therefore by substituting the GARCH(1,1) model in the quadratic equation (30) above, we have the expression below,

$$\sum_{i=1}^{T} \varepsilon_{t-1}^{2} \left(\omega + \alpha\varepsilon_{t-1}^{2} + \beta\sigma_{t-1}^{2}\right) - S\sigma_t - V = 0$$

$$\omega\sum_{i=1}^{T} \varepsilon_{t-1}^{2} + \alpha\sum_{i=1}^{T} (\varepsilon_{t-1}^{2})^2 + \beta\sum_{i=1}^{T} \varepsilon_{t-1}^{2}\,\sigma_{t-1}^{2} - S\sigma_t - V = 0 \tag{31}$$

Also, substituting for omega in equation (31) with equation (27), yields the following

$$\sum_{i=1}^{T} \varepsilon_{t-1}^{2} \left(\frac{D\sigma_t + H - \alpha\sum_{i=1}^{T}\varepsilon_{t-1}^{2} - \beta\sum_{i=1}^{T}\sigma_{t-1}^{2}}{T}\right) + \alpha\sum_{i=1}^{T} (\varepsilon_{t-1}^{2})^2 + \beta\sum_{i=1}^{T} \varepsilon_{t-1}^{2}\,\sigma_{t-1}^{2} - S\sigma_t - V = 0$$

Expanding the expression above gives the equation below

$$D\sigma_t\sum_{i=1}^{T} \varepsilon_{t-1}^{2} + H\sum_{i=1}^{T} \varepsilon_{t-1}^{2} + \alpha\left(\sum_{i=1}^{T} \varepsilon_{t-1}^{2}\right)^2 - \beta\sum_{i=1}^{T} \varepsilon_{t-1}^{2}\sum_{i=1}^{T} \sigma_{t-1}^{2} + \alpha T\sum_{i=1}^{T} (\varepsilon_{t-1}^{2})^2 + \beta T\sum_{i=1}^{T} \varepsilon_{t-1}^{2}\,\sigma_{t-1}^{2} - ST\sigma_t - VT = 0$$

Therefore, collecting like-terms and making alpha the subject formula gives the equation below,

$$\alpha = \frac{ST\sigma_t + VT - (D\sigma_t + H)\sum_{i=1}^{T}\varepsilon_{t-1}^{2} + \beta\left[\sum_{i=1}^{T}\varepsilon_{t-1}^{2}\sum_{i=1}^{T}\sigma_{t-1}^{2} - T\sum_{i=1}^{T}\varepsilon_{t-1}^{2}\,\sigma_{t-1}^{2}\right]}{\left\{T\sum_{i=1}^{T}(\varepsilon_{t-1}^{2})^2 - \left(\sum_{i=1}^{T}\varepsilon_{t-1}^{2}\right)^2\right\}} \tag{32}$$

Also, to estimating the parameter beta, we partially differentiate equation (20) with respect to beta, thus the expression below describes the procedure;

$$\frac{dl(.)}{d\beta} = \frac{dl(.)}{d\sigma_t^2} \times \frac{d\sigma_t^2}{d\beta}$$

Hence, by differentiating each term from the RHS of the above expression, we obtain the following expressions,

$$\frac{dl(.)}{d\beta} = \left\{-\frac{T}{2\sigma_t^2} + \frac{(pq+1)}{2q\sigma_t^2}\sum_{i=1}^{T} Zg(Z)\right\} \times \sigma_{t-1}^{2}$$

Therefore, by expanding and equating the expression above to zero, we obtain the following

$$\frac{dl(.)}{d\beta} = -\frac{T\varepsilon_{t-1}^{2}}{2\sigma_t^2} + \frac{(pq+1)}{2q\sigma_t^2}\sum_{i=1}^{T} \sigma_{t-1}^{2}\left\{m_i Z_{[i]} + n_i Z_{[i]}^{2}\right\} = 0$$

$$\frac{dl(.)}{d\alpha} = -T\sigma_{t-1}^{2} + \frac{(pq+1)}{q}\sum_{i=1}^{T} m_i\frac{\varepsilon_t\sigma_{t-1}^{2}}{\sigma_t} + \frac{(pq+1)}{q}\sum_{i=1}^{T} n_i\frac{\varepsilon_t^2\sigma_{t-1}^{2}}{\sigma_t^2} = 0$$

$$\frac{dl(.)}{d\alpha} = -\sigma_t^2 T\sigma_{t-1}^{2} + \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} m_i\varepsilon_t\sigma_{t-1}^{2}\right]\sigma_t + \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} n_i\varepsilon_t^2\sigma_{t-1}^{2}\right] = 0$$

$$\sigma_t^2\sum_{i=1}^{T} \sigma_{t-1}^{2} - \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} m_i\varepsilon_t\sigma_{t-1}^{2}\right]\sigma_t - \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} n_i\varepsilon_t^2\sigma_{t-1}^{2}\right] = 0$$

(33)

From equation (33) above,

Let $F = \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} m_i \varepsilon_t \sigma_{t-1}^2\right]$ and $G = \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} n_i \varepsilon_t^2 \sigma_{t-1}^2\right]$ (34)

Then equation (33) becomes,

$$\sigma_t^2 \sum_{i=1}^{T} \sigma_{t-1}^2 - F\sigma_t - G = 0 \tag{35}$$

Where F and G are as stated in equation (34).

Therefore by substituting the GARCH(1,1) model in the quadratic equation (35) above, we have the expression below

$$\omega \sum_{i=1}^{T} \sigma_{t-1}^2 \, \alpha \sum_{i=1}^{T} \varepsilon_{t-1}^2 \, \sigma_{t-1}^2 + \beta \sum_{i=1}^{T} (\sigma_{t-1}^2)^2 - F\sigma_t - G = 0 \tag{36}$$

Substituting for ω in equation (36) with equation (27) we obtain the following,

$$\sum_{i=1}^{T} \sigma_{t-1}^2 \left(\frac{D\sigma_t + H - \alpha \sum_{i=1}^{T} \varepsilon_{t-1}^2 - \beta \sum_{i=1}^{T} \sigma_{t-1}^2}{T} \right) + \alpha \sum_{i=1}^{T} \varepsilon_{t-1}^2 \, \sigma_{t-1}^2 + \beta \sum_{i=1}^{T} (\sigma_{t-1}^2)^2 - F\sigma_t - G = 0$$

Expanding the expression above yields the equation below,

$$(D\sigma_t + H) \sum_{i=1}^{T} \sigma_{t-1}^2 - \alpha \sum_{i=1}^{T} \varepsilon_{t-1}^2 \sum_{i=1}^{T} \sigma_{t-1}^2 - \beta \left(\sum_{i=1}^{T} \sigma_{t-1}^2 \right)^2 + \alpha T \sum_{i=1}^{T} \varepsilon_{t-1}^2 \, \sigma_{t-1}^2 + \beta T \sum_{i=1}^{T} (\sigma_{t-1}^2)^2 - FT\sigma_t - GT = 0 \tag{37}$$

From equation (32),

Let $I = \frac{ST\sigma_t + VT - (D\sigma_t + H)\sum_{i=1}^{T} \varepsilon_{t-1}^2}{\left\{T\sum_{i=1}^{T}(\varepsilon_{t-1}^2)^2 - \left(\sum_{i=1}^{T} \varepsilon_{t-1}^2\right)^2\right\}}$ and $U = \frac{\left[\sum_{i=1}^{T} \varepsilon_{t-1}^2 \sum_{i=1}^{T} \sigma_{t-1}^2 - T\sum_{i=1}^{T} \varepsilon_{t-1}^2 \sigma_{t-1}^2\right]}{\left\{T\sum_{i=1}^{T}(\varepsilon_{t-1}^2)^2 - \left(\sum_{i=1}^{T} \varepsilon_{t-1}^2\right)^2\right\}}$ (38)

Therefore, substituting for *I* and *U* in equation (32) yields,

$$\alpha = I + \beta U \tag{39}$$

Hence, collecting like-terms from equation (37) yields the expression below,

$$(D\sigma_t + H) \sum_{i=1}^{T} \sigma_{t-1}^2 + \beta \left\{ T \sum_{i=1}^{T} (\sigma_{t-1}^2)^2 - \left(\sum_{i=1}^{T} \sigma_{t-1}^2 \right)^2 \right\} - \alpha \left\{ \sum_{i=1}^{T} \varepsilon_{t-1}^2 \sum_{i=1}^{T} \sigma_{t-1}^2 - T \sum_{i=1}^{T} \varepsilon_{t-1}^2 \, \sigma_{t-1}^2 \right\} - FT\sigma_t - GT = 0$$

Therefore, substituting for alpha in the expression above with equation (39),

$$(D\sigma_t + H) \sum_{i=1}^{T} \sigma_{t-1}^2 + \beta \left\{ T \sum_{i=1}^{T} (\sigma_{t-1}^2)^2 - \left(\sum_{i=1}^{T} \sigma_{t-1}^2 \right)^2 \right\} - (I + \beta U) \left\{ \sum_{i=1}^{T} \varepsilon_{t-1}^2 \sum_{i=1}^{T} \sigma_{t-1}^2 + T \sum_{i=1}^{T} \varepsilon_{t-1}^2 \, \sigma_{t-1}^2 \right\} - FT\sigma_t - GT = 0$$

Expanding the expression above gives the expression below,

$$(D\sigma_t + H)\sum_{i=1}^{T}\sigma_{t-1}^2 + \beta\left\{T\sum_{i=1}^{T}(\sigma_{t-1}^2)^2 - \left(\sum_{i=1}^{T}\sigma_{t-1}^2\right)^2\right\} + \beta U\left\{\sum_{i=1}^{T}\varepsilon_{t-1}^2\sum_{i=1}^{T}\sigma_{t-1}^2 + T\sum_{i=1}^{T}\varepsilon_{t-1}^2\,\sigma_{t-1}^2\right\}$$
$$- I\left\{\sum_{i=1}^{T}\varepsilon_{t-1}^2\sum_{i=1}^{T}\sigma_{t-1}^2 + T\sum_{i=1}^{T}\varepsilon_{t-1}^2\,\sigma_{t-1}^2\right\} - FT\sigma_t - GT = 0$$

Therefore, collecting like-terms and making beta the subject formula of the expression above yields the equation below,

$$\beta = \frac{FT\sigma_t + GT - (D\sigma_t + H)\sum_{i=1}^{T}\sigma_{t-1}^2 + I\{\sum_{i=1}^{T}\varepsilon_{t-1}^2\sum_{i=1}^{T}\sigma_{t-1}^2 + T\sum_{i=1}^{T}\varepsilon_{t-1}^2\,\sigma_{t-1}^2\}}{\{T\sum_{i=1}^{T}(\sigma_{t-1}^2)^2 - (\sum_{i=1}^{T}\sigma_{t-1}^2)^2\} + U\{\sum_{i=1}^{T}\varepsilon_{t-1}^2\sum_{i=1}^{T}\sigma_{t-1}^2 + T\sum_{i=1}^{T}\varepsilon_{t-1}^2\,\sigma_{t-1}^2\}} \tag{40}$$

3.9 *Estimating the Parameters of TS-GARCH (1,1) Model*

The TS-GARCH (1,1) with reasons stated by eq(9), is expressed below;

$$\sigma_t = \omega + \alpha|\varepsilon_{t-1}| + \beta\sigma_{t-1}$$

Therefore, estimating their respective parameters is obtained by substituting the TS-GARCH(1,1) model into the variance of the generalized t distribution.

Firstly, estimating the parameters, we differentiating the ML function with respect to the standard deviation, then further differentiate with respect to desired parameter estimates, thus the expressions below describe the procedure;
Therefore, differentiating the ML function with respect to the standard deviation,

$$\frac{dl(.)}{d\sigma_t} = -\frac{T}{\sigma_t} + \frac{(pq+1)}{q\sigma_t}\sum_{i=1}^{T} Zg(Z) \tag{41}$$

And the function g(Z) above is:

$$g(Z_{[i]}) \cong m_i + n_i Z_{[i]}$$

Therefore, substituting the function g(Z) in equation (41) above yields,

$$\frac{dl(.)}{d\sigma_t} = -\frac{T}{\sigma_t} + \frac{(pq+1)}{q\sigma_t}\sum_{i=1}^{T} Z_{[i]}\{m_i + n_i Z_{[i]}\}$$

$$\frac{dl(.)}{d\sigma_t} = -\frac{T}{\sigma_t} + \frac{(pq+1)}{q\sigma_t}\sum_{i=1}^{T}\{m_i Z_{[i]} + n_i Z_{[i]}^2\}$$

$$\frac{dl(.)}{d\sigma_t} = -T + \frac{(pq+1)}{q}\sum_{i=1}^{T} m_i Z_{[i]} + \frac{(pq+1)}{q}\sum_{i=1}^{T} n_i Z_{[i]}^2$$

Therefore, the expression for the variance of the GTD for the TS-GARCH(1,1) is;

$$\frac{dl(.)}{d\sigma_t} = -T + \frac{(pq+1)}{q}\sum_{i=1}^{T} m_i \frac{\varepsilon_t}{\sigma_t} + \frac{(pq+1)}{q}\sum_{i=1}^{T} n_i \frac{\varepsilon_t^2}{\sigma_t^2}$$

$$\frac{dl(.)}{d\sigma_t} = -T(\sigma_t)^2 + \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} m_i \varepsilon_t\right]\sigma_t + \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} n_i \varepsilon_t^2\right]$$

Hence, equating the expression above to zero, the following expression is obtained;

$$T(\sigma_t)^2 - \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} m_i \varepsilon_t\right]\sigma_t - \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} n_i \varepsilon_t^2\right] = 0$$

$$T(\sigma_t)^2 - D\sigma_t - H = 0 \tag{42}$$

where T is the number of observations, D and H are as stated in equation (23).
To estimating the parameter omega, we differentiate equation (41) with respect to omega, thus the procedure below;

$$\frac{dl(.)}{d\omega} = \frac{dl(.)}{d\sigma_t} \times \frac{d\sigma_t}{d\omega}$$

where $\frac{d\sigma_t}{d\omega} = 1$
Therefore substituting the TS-GARCH model in the equation (42) yields the following,

$$T(\omega + \alpha|\varepsilon_{t-1}| + \beta\sigma_{t-1})^2 - D\sigma_t - H = 0$$

$$T(\omega + \alpha|\varepsilon_{t-1}| + \beta\sigma_{t-1}) = \sqrt{(D\sigma_t + H)}$$

Therefore, expanding the expression above and making omega the subject formula gives the equation below;

$$\omega_{TS} = \frac{\sqrt{(D\sigma_t + H)} - \alpha\sum_{i=1}^{T}|\varepsilon_{t-1}| - \beta\sum_{i=1}^{T}\sigma_{t-1}}{T} \tag{43}$$

Estimating the parameter alpha, we further differentiate equation (41) with respect to alpha, thus the expression below describes the procedure;

$$\frac{dl(.)}{d\alpha} = \frac{dl(.)}{d\sigma_t} \times \frac{d\sigma_t}{d\alpha}$$

Hence, by differentiating each term from the RHS of the above expression, we obtain the following expressions,

$$\frac{dl(.)}{d\alpha} = \left\{-\frac{T}{\sigma_t} + \frac{(pq+1)}{q\sigma_t}\sum_{i=1}^{T} Zg(Z)\right\} \times |\varepsilon_{t-1}|$$

Therefore, by expanding and equating the expression above to zero, we obtain the following,

$$(\sigma_t)^2\sum_{i=1}^{T}|\varepsilon_{t-1}| - \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} m_i\varepsilon_t|\varepsilon_{t-1}|\right]\sigma_t - \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} n_i\varepsilon_t^2|\varepsilon_{t-1}|\right] = 0 \quad (44)$$

From equation (44)

Let $S_{TS} = \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} m_i\varepsilon_t|\varepsilon_{t-1}|\right]$ and $V_{TS} = \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} n_i\varepsilon_t^2|\varepsilon_{t-1}|\right]$ (45)

Therefore, equation (44) becomes,

$$(\sigma_t)^2\sum_{i=1}^{T}|\varepsilon_{t-1}| - S_{TS}\sigma_t - V_{TS} = 0 \quad (46)$$

Where S_{TS} and V_{TS} are as stated by equation (45)

Therefore by substituting the TS-GARCH(1,1) model in the quadratic equation (46) above, we have the expression below,

$$\sum_{i=1}^{T}|\varepsilon_{t-1}|\,(\omega + \alpha|\varepsilon_{t-1}| + \beta\sigma_{t-1})^2 - S_{TS}\sigma_t - V_{TS} = 0$$

$$\sum_{i=1}^{T}|\varepsilon_{t-1}|\,(\omega + \alpha|\varepsilon_{t-1}| + \beta\sigma_{t-1}) = \sqrt{(S_{TS}\sigma_t + V_{TS})}$$

$$\omega\sum_{i=1}^{T}|\varepsilon_{t-1}| + \alpha\sum_{i=1}^{T}(|\varepsilon_{t-1}|)^2 + \beta\sum_{i=1}^{T}|\varepsilon_{t-1}|\,\sigma_{t-1} = \sqrt{(S_{TS}\sigma_t + V_{TS})}$$

Hence, substituting for omega in the expression above with equation (43) and expanding the resulting expression, making alpha the subject formula yields the following equation,

$$\alpha_{TS} = \frac{T\sqrt{(S_{TS}\sigma_t + V_{TS})} - \sqrt{(D\sigma_t + H)}\sum_{i=1}^{T}|\varepsilon_{t-1}| + \beta[\sum_{i=1}^{T}|\varepsilon_{t-1}|\sum_{i=1}^{T}\sigma_{t-1} - T\sum_{i=1}^{T}|\varepsilon_{t-1}|\,\sigma_{t-1}]}{\{T\sum_{i=1}^{T}(|\varepsilon_{t-1}|)^2 - (\sum_{i=1}^{T}|\varepsilon_{t-1}|)^2\}} \quad (47)$$

From equation (47),

Let $I_{TS} = \frac{T\sqrt{(S_{TS}\sigma_t+V_{TS})}-\sqrt{(D\sigma_t+H)}\sum_{i=1}^{T}|\varepsilon_{t-1}|}{\left\{T\sum_{i=1}^{T}(|\varepsilon_{t-1}|)^2-(\sum_{i=1}^{T}|\varepsilon_{t-1}|)^2\right\}}$ and $U_{TS} = \frac{[\sum_{i=1}^{T}|\varepsilon_{t-1}|\sum_{i=1}^{T}\sigma_{t-1}-T\sum_{i=1}^{T}|\varepsilon_{t-1}|\sigma_{t-1}]}{\left\{T\sum_{i=1}^{T}(|\varepsilon_{t-1}|)^2-(\sum_{i=1}^{T}|\varepsilon_{t-1}|)^2\right\}}$ (48)

Therefore, equation (47) becomes,

$$\alpha_{TS} = I_{TS} + \beta_{TS} \quad (49)$$

Also, to estimating the parameter beta, we partially differentiate equation (41) with respect to beta, thus the expression below describes the procedure;

$$\frac{dl(.)}{d\beta} = \frac{dl(.)}{d\sigma_t}\times\frac{d\sigma_t}{d\beta}$$

Hence, by differentiating each term from the RHS of the above expression, we obtain the following expressions,

$$\frac{dl(.)}{d\beta} = \left\{ -\frac{T}{\sigma_t} + \frac{(pq+1)}{q\sigma_t} \sum_{i=1}^{T} Zg(Z) \right\} \times \sigma_{t-1}$$

Therefore, by expanding and equating the expression above to zero, we obtain the following,

$$(\sigma_t)^2 \sum_{i=1}^{T} \sigma_{t-1} - \left[\frac{(pq+1)}{q} \sum_{i=1}^{T} m_i \varepsilon_t \sigma_{t-1} \right] \sigma_t - \left[\frac{(pq+1)}{q} \sum_{i=1}^{T} n_i \varepsilon_t^2 \sigma_{t-1} \right] = 0 \tag{50}$$

From equation (50)

Let $F_{TS} = \left[\frac{(pq+1)}{q} \sum_{i=1}^{T} m_i \varepsilon_t \sigma_{t-1} \right]$ and $G_{TS} = \left[\frac{(pq+1)}{q} \sum_{i=1}^{T} n_i \varepsilon_t^2 \sigma_{t-1} \right]$ (51)

Therefore, equation (50) becomes

$$(\sigma_t)^2 \sum_{i=1}^{T} \sigma_{t-1} - F_{TS}\sigma_t - G_{TS} = 0 \tag{52}$$

where F_{TS} and G_{TS} are stated by equation (51)

Therefore by substituting the TS-GARCH(1,1) model in the quadratic equation (52) above, we have the expressions below,

$$\sum_{i=1}^{T} \sigma_{t-1} \left(\omega + \alpha|\varepsilon_{t-1}| + \beta\sigma_{t-1}\right)^2 - F_{TS}\sigma_t - G_{TS} = 0$$

$$\sum_{i=1}^{T} \sigma_{t-1} \left(\omega + \alpha|\varepsilon_{t-1}| + \beta\sigma_{t-1}\right)^2 = \sqrt{(F_{TS}\sigma_t + G_{TS})}$$

$$\omega \sum_{i=1}^{T} \sigma_{t-1} + \alpha \sum_{i=1}^{T} |\varepsilon_{t-1}|\sigma_{t-1} + \beta \sum_{i=1}^{T} (\sigma_{t-1})^2 = \sqrt{(F_{TS}\sigma_t + G_{TS})}$$

Therefore, inserting equation (43) for omega, equation (47) for alpha in the expression above, we further expand the resulting equation, hence making beta the subject formula gives the equation below,

$$\beta_{TS} = \frac{T\sqrt{(F_{TS}\sigma_t + G_{TS})} - \sqrt{(D\sigma_t + H)} \sum_{i=1}^{T} \sigma_{t-1} + I_{TS}\left[\sum_{i=1}^{T} |\varepsilon_{t-1}| \sum_{i=1}^{T} \sigma_{t-1} - T\sum_{i=1}^{T} |\varepsilon_{t-1}|\, \sigma_{t-1}\right]}{\left\{T\sum_{i=1}^{T} (\sigma_{t-1})^2 - \left(\sum_{i=1}^{T} \sigma_{t-1}\right)^2\right\} + U_{TS}\left\{\sum_{i=1}^{T} |\varepsilon_{t-1}| \sum_{i=1}^{T} \sigma_{t-1} - T\sum_{i=1}^{T} |\varepsilon_{t-1}|\, \sigma_{t-1}\right\}} \tag{53}$$

	OMEGA	***ALPHA***	***BETA***
GARCH(1,1)	$\omega = \frac{D\sigma_t + H - \alpha\sum_{i=1}^{T}\varepsilon_{t-1}^2 - \beta\sum_{i=1}^{T}\sigma_{t-1}^2}{T}$	$\alpha = \frac{ST\sigma_t + VT - (D\sigma_t + H)\sum_{i=1}^{T}\varepsilon_{t-1}^2 + \beta\left[\sum_{i=1}^{T}\varepsilon_{t-1}^2\sum_{i=1}^{T}\sigma_{t-1}^2 - T\sum_{i=1}^{T}\varepsilon_{t-1}^2\sigma_{t-1}^2\right]}{\left\{T\sum_{i=1}^{T}(\varepsilon_{t-1}^2)^2 - \left(\sum_{i=1}^{T}\varepsilon_{t-1}^2\right)^2\right\}}$	$\beta = \frac{FT\sigma_t + GT - (D\sigma_t + H)\sum_{i=1}^{T}\sigma_{t-1}^2 + I\left(\sum_{i=1}^{T}\varepsilon_{t-1}^2\sum_{i=1}^{T}\sigma_{t-1}^2 + T\sum_{i=1}^{T}\varepsilon_{t-1}^2\sigma_{t-1}^2\right)}{\left\{T\sum_{i=1}^{T}(\sigma_{t-1}^2)^2 - \left(\sum_{i=1}^{T}\sigma_{t-1}^2\right)^2\right\} + U\left(\sum_{i=1}^{T}\varepsilon_{t-1}^2\sum_{i=1}^{T}\sigma_{t-1}^2 + T\sum_{i=1}^{T}\varepsilon_{t-1}^2\sigma_{t-1}^2\right)}$
TS-GARCH(1,1)	$\omega_{TS} = \frac{\sqrt{(D\sigma_t + H)} - \alpha\sum_{i=1}^{T}\lvert\varepsilon_{t-1}\rvert - \beta\sum_{i=1}^{T}\sigma_{t-1}}{T}$	$\alpha_{TS} = \frac{T\sqrt{(S_{TS}\sigma_t + V_{TS})} - \sqrt{(D\sigma_t + H)}\sum_{i=1}^{T}\lvert\varepsilon_{t-1}\rvert + \beta\left[\sum_{i=1}^{T}\lvert\varepsilon_{t-1}\rvert\sum_{i=1}^{T}\sigma_{t-1} - T\sum_{i=1}^{T}\lvert\varepsilon_{t-1}\rvert\sigma_{t-1}\right]}{\left\{T\sum_{i=1}^{T}(\lvert\varepsilon_{t-1}\rvert)^2 - \left(\sum_{i=1}^{T}\lvert\varepsilon_{t-1}\rvert\right)^2\right\}}$	$\beta_{TS} = \frac{T\sqrt{(F_{TS}\sigma_t + G_{TS})} - \sqrt{(D\sigma_t + H)}\sum_{i=1}^{T}\sigma_{t-1} + I_{TS}\left[\sum_{i=1}^{T}\lvert\varepsilon_{t-1}\rvert\sum_{i=1}^{T}\sigma_{t-1} - T\sum_{i=1}^{T}\lvert\varepsilon_{t-1}\rvert\sigma_{t-1}\right]}{\left\{T\sum_{i=1}^{T}(\sigma_{t-1})^2 - \left(\sum_{i=1}^{T}\sigma_{t-1}\right)^2\right\} + U_{TS}\left\{\sum_{i=1}^{T}\lvert\varepsilon_{t-1}\rvert\sum_{i=1}^{T}\sigma_{t-1} - T\sum_{i=1}^{T}\lvert\varepsilon_{t-1}\rvert\sigma_{t-1}\right\}}$

where; T = Number of observations

$$D = \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} m_i \varepsilon_t\right] \quad H = \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} n_i \varepsilon_t^2\right] \quad S = \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} m_i \varepsilon_t \varepsilon_{t-1}^2\right] \quad V = \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} n_i \varepsilon_t^2 \varepsilon_{t-1}^2\right] \quad F = \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} m_i \varepsilon_t \sigma_{t-1}^2\right] \quad G = \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} n_i \varepsilon_t^2 \sigma_{t-1}^2\right],$$

$$I = \frac{ST\sigma_t + VT - (D\sigma_t + H)\sum_{i=1}^{T}\varepsilon_{t-1}^2}{\left\{T\sum_{i=1}^{T}\left(\varepsilon_{t-1}^2\right)^2 - \left(\sum_{i=1}^{T}\varepsilon_{t-1}^2\right)^2\right\}},\ U = \frac{\left[\sum_{i=1}^{T}\varepsilon_{t-1}^2\sum_{i=1}^{T}\sigma_{t-1}^2 - T\sum_{i=1}^{T}\varepsilon_{t-1}^2\sigma_{t-1}^2\right]}{\left\{T\sum_{i=1}^{T}\left(\varepsilon_{t-1}^2\right)^2 - \left(\sum_{i=1}^{T}\varepsilon_{t-1}^2\right)^2\right\}} \quad I_{TS} = \frac{T\sqrt{(S_{TS}\sigma_t + V_{TS})} - \sqrt{(D\sigma_t + H)}\sum_{i=1}^{T}|\varepsilon_{t-1}|}{\left\{T\sum_{i=1}^{T}(|\varepsilon_{t-1}|)^2 - \left(\sum_{i=1}^{T}|\varepsilon_{t-1}|\right)^2\right\}}$$

$$U_{TS} = \frac{\left[\sum_{i=1}^{T}|\varepsilon_{t-1}|\sum_{i=1}^{T}\sigma_{t-1} - T\sum_{i=1}^{T}|\varepsilon_{t-1}|\sigma_{t-1}\right]}{\left\{T\sum_{i=1}^{T}(|\varepsilon_{t-1}|)^2 - \left(\sum_{i=1}^{T}|\varepsilon_{t-1}|\right)^2\right\}}, \quad S_{TS} = \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} m_i \varepsilon_t |\varepsilon_{t-1}|\right],\ V_{TS} = \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} n_i \varepsilon_t^2 |\varepsilon_{t-1}|\right]$$

$$F_{TS} = \left[\frac{(pq+1)}{q}\sum_{i=1}^{T} m_i \varepsilon_t \sigma_{t-1}\right].$$

4.0 Analysis and Results

Data used in this study are the daily return data of the Nigerian Naira /US dollar exchange rate for the period August 1, 1995 to June 1, 2017, obtained from the 2017 annual statistical bulletin of the Central Bank of Nigeria. R-package was used to carry out the analysis of collected data. The time plot of the series is presented by Figure 1.

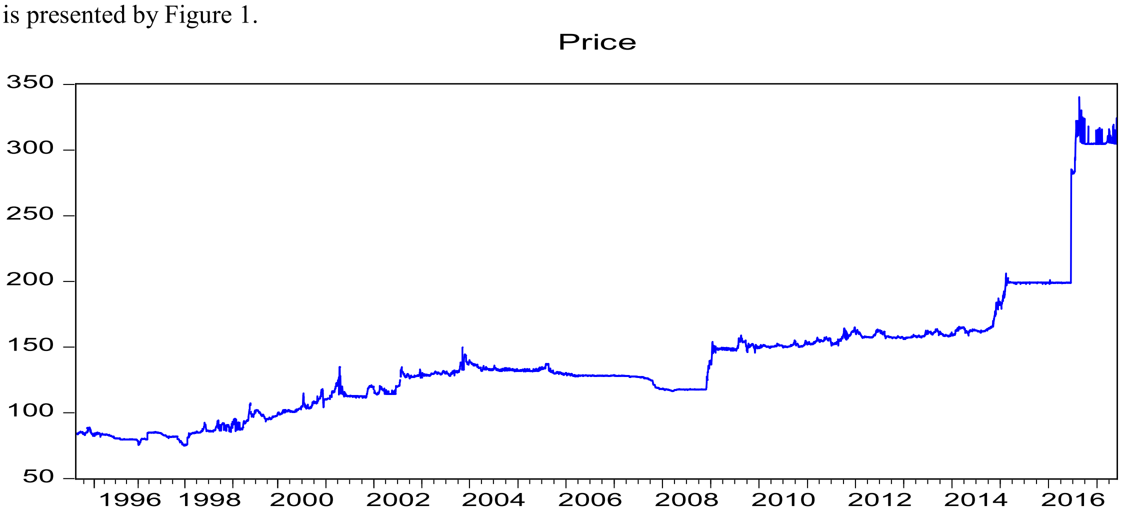


Figure 1: Plot of Foreign Exchange Price of Naira to US Dolla from August 1995 to June 2017.

4.1 *Descriptive Statistics*

Table 4.1 presents the descriptive statistics of the return series of the daily foreign exchange rate between the US Dollar/Nigerian Naira. From the table, the expected return(mean) for the series is positive, this implies that the Naira has witnessed more depreciation than appreciations against the US Dollar over the period under consideration.

The standard deviation is also larger than the expected return. The large standard deviation for the Dollar/Naira exchange rate indicates that the rate is less stable (more volatile or highly risky). Considering the distribution of the return series, the positive skewness (a right tail) of the return series relative to the normal distribution (0 for the normal distribution) indicate a higher possibility of depreciation of the Naira. Also, the kurtosis value shows that the return series is leptokurtic(exhibit fat tail) in nature. This implies that the return series have a substantial peak in the distribution, an indication of non-normality.

The Jarque-Bera normality test statistics (JB test) are also significantly large for the return series as shown in Table 4.4, these indicate that the return series is not normally distributed.

Table 4.1: Summary Statistics

Mean	Median	Mini.	Max.	Std. Dev.	Skewness	Kurtosis	Jarque-Bera	Prob.
0.0229	0.0000	-15.4151	34.9891	1.0505	6.2341	251.8611	13823.1534	2.2e-16

4.2 *Stationarity and Heteroscedasticity Test*

The ADF statistic test presented in Table 4.2 tests the null hypothesis of unit root against the alternative hypothesis of no unit root. The decision rule is to reject the null hypothesis when the value of the test statistic is less than the critical value. The ADF test statistic is greater than all the critical values in absolute terms, so the hypothesis of non-stationarity is rejected.

Table 4.2: Unit Root Test (ADF test statistic)

	t-Statistic	Level	Critical Values	Prob.*
Return Series	-12.94825	1%	-3.431337	< 0.0001
		5%	-2.861861	
		10%	-2.566983	

The results of the heteroscedasticity test for the return series are given in Table 4.3. The null hypothesis of no ARCH effect is rejected for the return series.

Table 4.3: Testing for ARCH Effects

ARCH LM-test	Lag 2	Lag 5	Lag 10
Statistics	495.11	511.5	17.909
P-Value	2.2e-16	2.2e-16	0.05652

Furthermore, the Ljung Box Q-Statistic for squared residuals shows that higher-order serial correlation is significant.

Table 4.4: Ljung Box Q-Statistic

Ljung Box Q-Statistic	LB-Q(10)	LB-Q^2 (10)
Statistics	752.46	1312.3
P-Value	2.2e-16	2.2e-16

The estimates of the parameters of the GARCH models are given in Table 4.5. For all fitted models, the estimates are found to be significant at 1%, 5% and 10% significant levels for the return series. The significance of these estimates shows the importance of modeling extreme and unusual market events that have occurred during the sample period (Zivot, 2009).

The significance of both a and $ß$ across the various models indicate that news about volatility (i.e. fluctuation) from the previous periods have explanatory power on current volatility. According to Longmore and Robinson (2004), positive values for these coefficients suggests that as the market approaches expected future rate, volatility will tend to increase. In models, like the TS-GARCH with the

GTD and Normal distribution respectively, the coefficients of a and β were high (>1)while at other times, like the GARCH with the GTD and Normal distribution respectively, they are relatively lower in comparison.

Table 4.5: Parameter estimates of GARCH Models

Models	Equations	Parameters	Generalized T-Distribution		Normal Distribution	
			Coefficient	**P-Value**	**Coefficient**	**P-Value**
GARCH (1,1)	Mean	MA (Θ)	-0.3877	0.0008	-0.3651	0.0008
	Variance	Intercept(ω)	-0.0956	0.0003	-0.0829	0.0002
		ARCH(α)	0.2429	0.0014	0.1089	0.0007
		GARCH(β)	0.5692	0.0018	0.4943	0.0011
Persistence			0.8120		0.6032	
Log-Likelihood			-386106.5		-407831.8	
AIC			772221		815671.7	
BIC			772240.7		815691.3	
TS-GARCH (1,1)	Mean	MA (Θ)	-0.3413	0.0008	-0.3196	0.0008
	Variance	Intercept(ω)	-0.1846	0.0006	-0.1733	0.0004
		ARCH(α)	0.3243	0.0011	0.2026	0.0008
		GARCH(β)	1.0605	0.0023	1.0102	0.0016
Persistence			1.3848		1.2128	
Log-Likelihood			-355364.7		-379626.4	
AIC			710737.4		759260.7	
BIC			710757		759280.4	

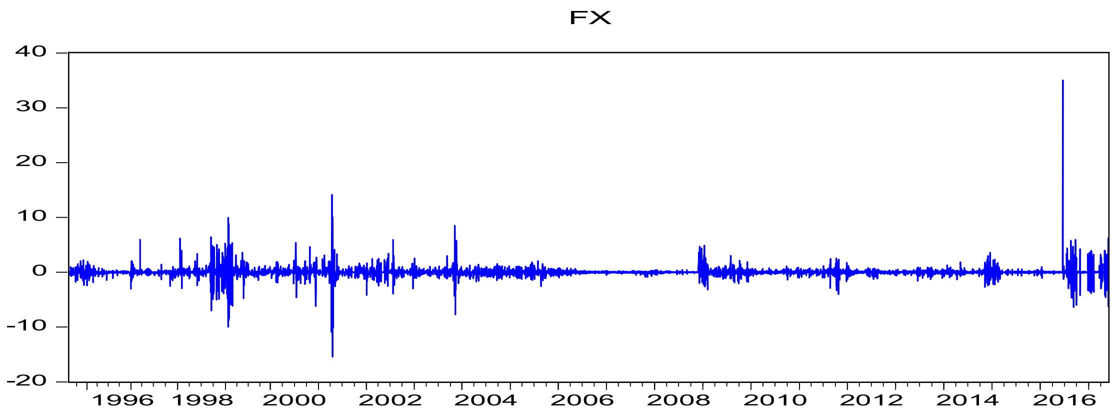


Figure 2: Plot of Foreign Exchange Returns

The high value of β as produced by the TS-GARCH(1,1) under the distributions considered indicates that shocks to the conditional variance are persistent while high value of a like in the EGARCH(1,1) model indicates that volatility adjusts quickly to changes in the market. The significance of a depicted by TS-GARCH(1,1) models appears to show the presence of volatility clustering in the models. Also, conditional volatility for these models tends to rise (fall) when the absolute value of the standardized residuals is large (smaller).

The estimated persistence coefficient ($a + \beta$) for the GARCH and TS-GARCH for all the return time series is such that if the sum is less than 1, as observed in the GARCH(1,1), shows persistent volatility in the model and is required to have a mean-reverting process, that returns eventually move back toward the mean or average over time, otherwise, if the sum is greater than 1, as in the TS-GARCH(1,1), it indicates that shocks to volatility are very high and will remain high as the variances are not stationary under the models.

Using the model selection criteria, the best volatility model for the return series is observed to be the TS-GARCH(1,1) with GTD, because it has a minimum value for the selection criteria when compared to other models under the GTD followed by the GARCH(1,1) model. The plot of the foreign exchange returns is presented in Figure 2.

5.0 Discussion and Conclusion

This study focused on modeling volatility with the applicability of first-order GARCH family models, that is, GARCH(1,1), TS-GARCH(1,1) – a fat tail alongside the generalized t-distribution for error innovation, as well as the normal distribution for performance comparison.

Using the daily data of the Naira/Dollar exchange rate to model the volatility of exchange rate returns, the GARCH (1,1), TS-GARCH (1,1) with generalized t-error distribution produced better estimates when compared to the corresponding estimates produced by the respective models with the normal distribution. This conclusion made is based on the value of the Information Criterion used in the study.

Overall, the TS-GARCH (1,1) model with the generalized t-error distribution is identified as the best performing model given that its information Criterion is the smallest when compared to the other models. In essence, this model will capture all the necessary stylize facts (common features) of the financial data, such as persistent, volatility clustering and asymmetric effects.

Reference

Arslan, O. and Genc, A. I. (2003). Robust Location and Scale Estimation Based on the Univariate Generalized t Distribution. *Communications in Statistics*, 1505-1525.

Bera, A. K. and Higgins, M. L. (1993). On ARCH Models: Properties, Estimation, and Testing. *Journal of Economic Surveys*, 305 – 362.

Bollerslev, T. (1986). Generalized Autoregressive Conditional Heteroscedasticity. *Journal of Econometrics*, 307-327.

Bollerslev, T., Engle R. F. and Nelson, B. D. (1994). ARCH Models. *Handbook of Econometrics*, Chapter 49.

Ding, Z. , Engle, R.F. and Granger, C.W.J. (1993). Long Memory Properties of Stock Market Returns and a New Model. *Journal of Empirical Finance*, 83 – 106.

Engle, R. F. and Ng, K. V. (1993). Measuring and Testing the Impact of News on Volatility. *Journal of Finance*, 1749-1778.

Engle, R. F. (1982). Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of the United Kingdom Inflation. *Econometrica*, 987-1008.

Escanciano C. J. and Lobato I. N. (2009). Testing the Martingale Hypothesis. In.K. Patterson and T. C. Mills (Eds.). *Palgrave Handbook of Econometrics*, 972-1003.

Glosten, L., Jagannathan, R. and Runkle, D. (1993). On the relationship between the Expected Value and the Volatility of the Nominal Excess Return on Stocks. *Journal of Finance*, 1779-1802.

McDonald J.B. and Newey W.K. (1988). Partially Adaptive Estimation of Regression Models via the Generalized t Distribution. *Econometric Theory*, 428-457.

Nelson, D. B. (1991). Conditional Heteroscedasticity in Asset Returns: A New approach. *Econometrica*, 347–70.

Olowe, R. A. (2009). Modelling Naira/Dollar Exchange Rate Volatility: Application of GARCH and Asymmetric Models. *International Review of Business Research Papers*, 377 – 398.

Poon, S. (2005). *A Practical Guide to Forecasting Financial Market Volatility.* Chichester: John Wiley & Sons Ltd.

Said, S.E. and Dickey, D. (1984). Testing for Unit Roots in Autoregressive Moving-Average Models with Unknown Order.

Biometrika, 599-607.

Schwert, W. (1990). Stock Volatility and Crash of '87. *Review of Financial Studies*, 77–102.

Taylor, S. (1986). *Modelling Financial Time Series.* Great Britain: John Wiley & Sons.

Tiku, M. (1967). Estimating the mean and standard deviation from censored normal samples. *Biometrika*, 155–165.

Tiku, M. (1968). Estimating the parameters of normal and logistic distributions from censored samples. *Journal of Statist.* , 64-74.

Tiku, M. L., Wong, W. K.,and Bian, G. (1999). Estimating parameters in autoregressive models in non-normal situations; symmetric innovations. *Commun. Statist. Theory Meth*, 315-341.

Tiku, M.L. and Suresh, R.P. (1992). A new method of estimation for location and scale parameters. *Journal of Stat. Plann. and Inf*, 281-292.

Tiku, M.L. (1967). Estimating the mean and standard deviation from a censored normal sample. *Biometrika*, 155-65.

Yoon, S. and Lee, K. S. (2008). The Volatility and Asymmetry of Won/Dollar Exchange Rate. *Journal of Social Sciences*, 7-9.

Zakoian, J. M. (1994). Threshold Heteroskedastic Models. *Journal of Economic Dynamics and Control*, 931-944.

Zivot, E. (2009). Practical Issues in the Analysis of Univariate GARCH Models. *Handbook of Financial Time Series*, 113–153.