A New Generalized Odd Generalized Exponentiated Skew-t Distribution for GARCH Models: Inference and Applications

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***Abstract* - In this research, a new generalized odd generalized exponentiated skew-t (GOGE$_{ST}$) innovation density for generalized autoregressive conditional heteroskedasticity (GARCH) models is proposed. Structural properties of the proposed distribution such as density and distribution functions, quantile function and raw moments were derived. To estimate the parameters of the proposed distribution via a simulation study, the maximum likelihood estimation criterion was used. Application on Bitcoin log-returns was carried out to inspect the performance of the asymmetric GARCH models with GOGE$_{ST}$ innovation density relative to five innovation densities in volatility modeling. The research results showed that the asymmetric GARCH models under GOGE$_{ST}$ innovation density performed best for the Bitcoin log-returns conditional variance. More so, in the out-of-sample performance of the models, the threshold GARCH-GOGE$_{ST}$ model outperformed the other models in terms of superior volatility predictive ability.**

Keywords: *Asymmetric volatility models, Hybridized distributions, Simulation study, Bitcoin, Innovations.*

I. Introduction

In financial time series, it is widely known that asset returns exhibit some stylized features such as excess kurtosis, skewness, heavy tail, volatility clustering and leverage effects. This often results in overestimation or underestimation of actual volatility estimates. Thus, skewness and leptokurtic behaviours of assets returns have been modeled using heavy-tailed and skewed distributions (Nelson, 1991; Altun *et al.,* 2018). Several authors have investigated the performance of the GARCH-type models under heavy-tailed distributions like the skew Student-t, normal inverse gaussian, skew generalized error, Student-t, exponentiated half-logistic skew-t and generalized error distribution in estimation and prediction of asset returns volatility (Ural and Demireli, 2019; Samson *et al.,* 2020a; Samson *et al.,* 2020b; Yelamanchili, 2020; Gyamerah, 2019; Maree *et al.,* 2017; Kosapattarapim, 2013; Nagamani and Tripathy, 2018; Ngunyi *et al.,* 2019; Adubisi *et al.,* 2022a; Adubisi *et al.,* 2022b). Financial asset volatility is a key concern in portfolio allocation, risk management, investment decisions, option pricing, microeconomic and policy making. In this context, volatility predictions can play the role of a trigger factor for the financial markets as well as economy (Poon & Granger, 2003). Hence, modeling of asset return volatility is important in the field of finance given that volatility is widely used as the most important factor in financial assets investment (Mandelbrot, 1963).

Harvey and Sucarrat (2013) introduced the EGARCH model under the beta-skew Student-t density for accurate prediction of daily volatility. Agboola *et al.* (2018) estimated daily volatility for stock index returns by using a new generalization of the skew Student-t distribution and showed that it outperformed the skew normal, skew Student-t and skew generalized error densities. Moreover, Agboola *et al.* (2019) used the same distributional assumption for the S&P 500 stock returns and supported the use of skewed Student-t for asymmetric power ARCH model, since it has the lowest forecast performance measures. Harvey and Chakravarty (2008) and Harvey (2013) proposed the use of the beta-student-t distribution for the exponential GARCH (EGARCH) model in the estimation of volatility and able to capture the dynamics compared to the existing distributions. The Bayesian analysis of a stochastic model with generalized hyperbolic skew Student-t distribution with an efficient Markov chain Monte Carlo estimation method was considered using simulated and real data (Nakajima & Omori, 2012). Nelson (1991) resolved that volatility models which do not permit asymmetries in the conditional variance usually produce poor volatility forecast. Liu and Hung (2010) found that more flexible GARCH-type models are quite adequate in volatility prediction for all densities assumptions. As concerns the choice of the innovation distribution, Feng and Shi (2017) found that the leptokurtic distributions in GARCH-type models aided in generating more accurate volatility forecasts. However, development of robust distributions is still vital in increasing the accuracy of the monetary risk system.

The motivation in developing the new model is to create a more flexible distribution with symmetric, right and left-skewed, and unimodal features. The aim of this research is to propose a new conditional innovation density to produce more accurate volatility forecasts than the existing commonly used innovation densities in GARCH volatility models. For this reason, the generalized odd generalized exponentiated skew-t ($GOGE_{ST}$) distribution is introduced and it provides a unimodal, skewed and fat-tail shapes for the probability density function (pdf). Also, a new dynamic asymmetric GARCH-$GOGE_{ST}$ model for predicting daily volatility is introduced based on various asymmetric GARCH-type volatility model with $GOGE_{ST}$ innovation density given that the existing innovation densities are quite incapable of accounting for excess kurtosis and skewness in financial datasets.

This article is structured as follows: Section 2 presents some structural properties of the $GOGE_{ST}$ distribution. The GARCH-type models with existing and new conditional innovation distributions are presented in Section 3. Model selection and forecast evaluation measures are given in Section 4. Section 5 presents the empirical results of both estimation and prediction while conclusion in Section 6.

II THE GENERALIZED ODD GENERALIZED EXPONENTIATED SKEW-T DISTRIBUTION

Many cases in real-life situations do not conform to the assumption of normality. Hence, the interest of so many researchers have been aroused in developing flexible distributions which will serve as substitute to the normal distribution for modeling datasets with excess kurtosis and skewness characteristics. The generalized odd generalized exponential (GOGE) family of distributions was introduced by Alizadeh *et al.* (2017). The cumulative distribution function (cdf) is given by

$$F(y;\varphi,\tau)=\left\{1-e^{\frac{-G(y)^{\varphi}}{1-G(y)^{\varphi}}}\right\}^{\tau} \qquad (1)$$

and the corresponding pdf is given by

$$f(y;\varphi,\tau)=\frac{\varphi\tau g(y)G(y)^{\varphi-1}}{\left[1-G(y)^{\varphi}\right]^{2}}e^{\frac{-G(y)^{\varphi}}{1-G(y)^{\varphi}}}\left\{1-e^{\frac{-G(y)^{\varphi}}{1-G(y)^{\varphi}}}\right\}^{\tau-1} \qquad (2)$$

where $G(y)$ and $g(y)$ are the cdf and pdf of the baseline distribution. $\varphi,\tau>0$ are two additional shape parameters. In this research, the parent distribution is the skew student-t (ST) distribution introduced by Jones and Faddy (2003). The cdf of the skew-t (ST) distribution is given by

$$G_{ST}(y)=\frac{1}{2}\left(1+\frac{y}{\sqrt{\upsilon+y^{2}}}\right),\quad y\in\Re, \qquad (3)$$

and the corresponding pdf is given by

$$g_{ST}(y)=\frac{\upsilon}{2\left(\upsilon+y^{2}\right)^{3/2}},\quad y\in\Re \qquad (4)$$

where, υ is the parameter which controls the skewness. Now, inserting the function of Jones and Faddy (2003) into the function of Alizadeh *et al.* (2017). The cdf and pdf of the $GOGE_{ST}$ distribution are obtained, respectively, as follows:

$$F(y;\varphi,\tau)=\left\{1-e^{\frac{-\Psi(y)^{\varphi}}{1-\Psi(y)^{\varphi}}}\right\}^{\tau},\quad \varphi,\tau>0, y\in\Re \qquad (5)$$

$$f(y;\varphi,\tau)=\frac{\varphi\tau\psi(y)\Psi(y)^{\varphi-1}}{\left[1-\Psi(y)^{\varphi}\right]^{2}}e^{\frac{-\Psi(y)^{\varphi}}{1-\Psi(y)^{\varphi}}}\left\{1-e^{\frac{-\Psi(x)^{\varphi}}{1-\Psi(x)^{\varphi}}}\right\}^{\tau-1},\quad \varphi,\tau>0, y\in\Re \qquad (6)$$

Here, $\psi(y)$ and $\Psi(y)$ are the pdf and cdf of the ST distribution, respectively, $\varphi,\tau>0$ are the shape parameter and υ controls the skewness.

Figure 1 depicts the plots of the pdf for the $GOGE_{ST}$ distribution. As shown in Figure 1, the $GOGE_{ST}$ distribution offers new prospects to model unimodality, skewness, leptokurtic and heavy-tailed structures of most datasets from different fields. More so, the $GOGE_{ST}$ distribution is regarded as a good nominee in eliminating the lack of modeling capability of existing distributions in relations to accurate volatility prediction, since excess kurtosis and skewness are common features in most financial time series.

III MATHEMATICAL PROPERTIES OF THE $GOGE_{ST}$ DISTRIBUTION

3.1 Quantile Function

The quantile function $Q(U), U \in (0,1)$ is derived by inverting Equation (5). The quantile function of the GOGE$_{ST}$ distribution is given by:

$$Q(U) = \upsilon^{\frac{1}{2}} \left(\left\{ 2\left[\frac{-\log\left(1-u^{\frac{1}{\tau}}\right)}{\left(1-\log\left(1-u^{\frac{1}{\tau}}\right)\right)} \right]^{\frac{1}{\varphi}} \right\} - 1 \right) \Bigg/ \sqrt{1 - \left(\left\{ 2\left[\frac{-\log\left(1-u^{\frac{1}{\tau}}\right)}{\left(1-\log\left(1-u^{\frac{1}{\tau}}\right)\right)} \right]^{\frac{1}{\varphi}} \right\} - 1 \right)^2}, \qquad 0 < U < 1. \tag{7}$$

The quantile function is considered very useful in generating random variate from any continuous distribution. Therefore, the numerical values of the Median (M), skewness (S) and kurtosis (K) provided in Table 1, are simulated values of the median, skewness and kurtosis using the GOGE$_{ST}$ quantile function in Equation (7). The results show that the new model can handle negative and positive skewed, and leptokurtic datasets. The desity plots of the new GOGE$_{ST}$ at some selected parameter values is presented by Fig 1.

Table 1. Numerical values of some useful statistics

υ	φ	τ	**M**	**S**	**K**
0.5	0.8	0.7	-0.4351	-0.3173	-1.1428
	1.0	0.9	-0.1700	-0.1801	-0.5791
	1.2	1.0	-0.035	-0.1181	-0.3656
	1.7	1.3	0.207	-0.0327	-0.0999
	2.0	1.8	0.366	0.0019	0.0006
1.0	0.8	0.7	-0.6153	-0.3173	-1.1428
	1.0	0.9	-0.2399	-0.1801	-0.5792
	1.2	1.0	-0.050	-0.1181	-0.3656
	1.7	1.3	0.293	-0.0327	-0.0999
	2.0	1.8	0.518	0.0019	0.0006
2.0	0.8	0.7	-0.8701	-0.3173	-1.1428
	1.0	0.9	-0.3393	-0.1801	-0.5792
	1.2	1.0	-0.070	-0.1181	-0.3656
	1.7	1.3	0.415	-0.0327	-0.0999
	2.0	1.8	0.732	0.0019	0.0006

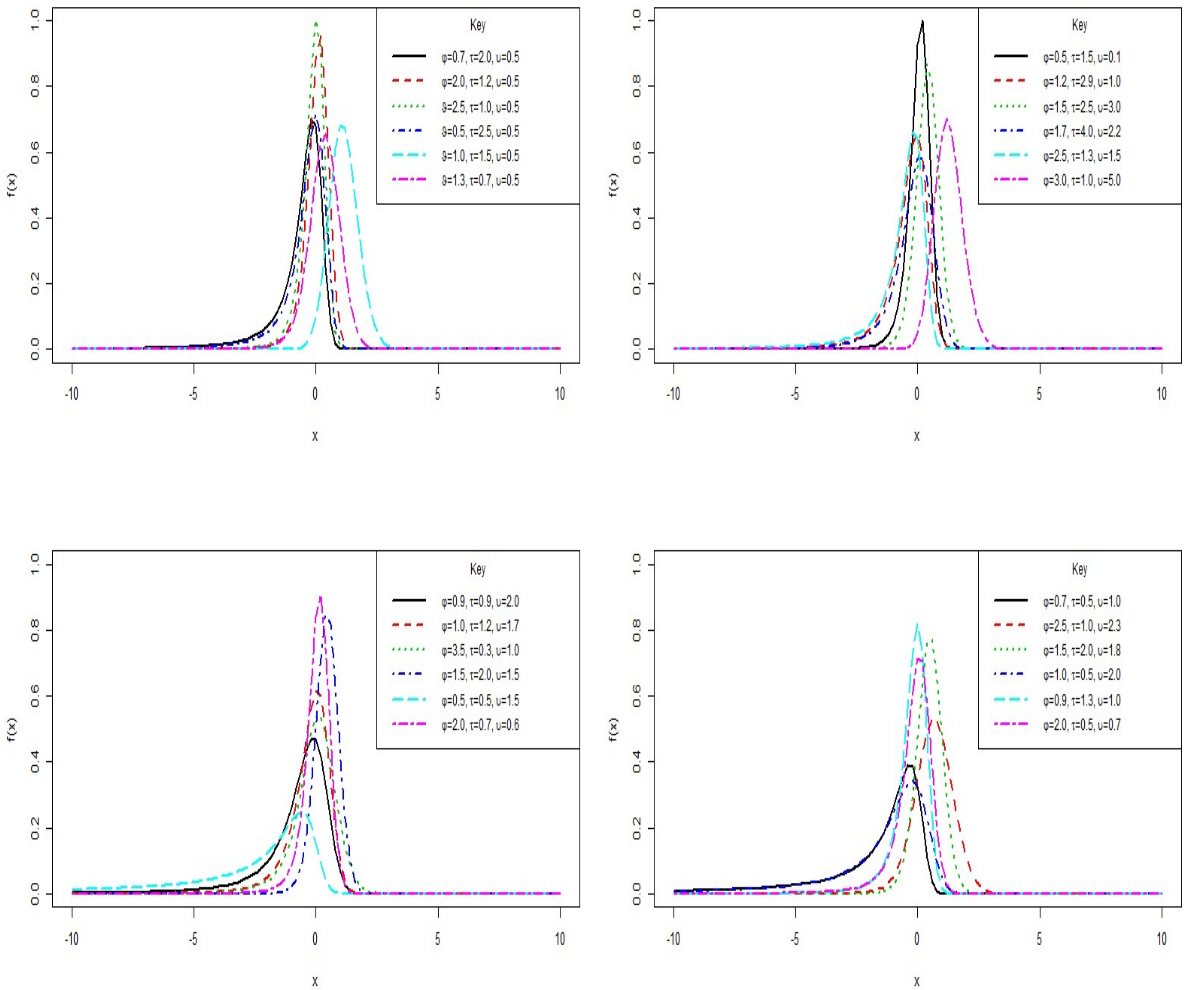

Figure 1: Density function (pdf) plots of the $GOGE_{ST}$ distribution for selected parameter values.

3.2 Linear representation

In this subsection, the linear representations of the probability density and distribution functions which are very useful in studying the mathematical properties of the $GOGE_{ST}$ distribution are derived. Firstly, the expansion of the PDF in Equation (5) is obtained by applying the generalized binomial series expansion, the density function of the $GOGE_{ST}$ distribution is expressed as

$$f(y) = \frac{\varphi \tau \upsilon \left[\frac{1}{2}\left(1+\frac{y}{\sqrt{\upsilon + y^2}}\right)\right]^{\varphi - 1}}{2\left(\upsilon + y^2\right)^{3/2}\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\upsilon+y^2}}\right)\right)^{\varphi}\right]^2} \sum_{i=0}^{\infty} (-1)^i \binom{\tau - 1}{i} e^{\frac{-(i+1)\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\upsilon+y^2}}\right)\right)^{\varphi}}{1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\upsilon+y^2}}\right)\right)^{\varphi}}} \tag{8}$$

But,

$$e^{\frac{-(i+1)\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\upsilon+y^2}}\right)\right)^{\varphi}}{1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\upsilon+y^2}}\right)\right)^{\varphi}}} = \sum_{j=0}^{\infty}(-1)^j\frac{(i+1)^j}{j!}\left(\frac{\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\upsilon+y^2}}\right)\right)^{\varphi}}{1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\upsilon+y^2}}\right)\right)^{\varphi}}\right)^j \tag{9}$$

Hence,

$$f(y)=\frac{\varphi\tau\upsilon}{2\left(\upsilon+y^2\right)^{3/2}}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(i+1)^j\left[\frac{1}{2}\left(1+\frac{y}{\sqrt{\upsilon+y^2}}\right)\right]^{\varphi(j+1)-1}}{j!\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\upsilon+y^2}}\right)\right)^{\varphi}\right]^{j+2}}\binom{\tau-1}{i} \tag{10}$$

Using the generalized binomial series, the PDF of the GOGE$_{ST}$ distribution can be defined as an infinite linear combination of the skew-t distribution expressed as

$$f(y)=\varphi\tau\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j+k}(i+1)^j}{j!}\binom{\tau-1}{i}\binom{-(j+2)}{k}\left(\frac{\upsilon}{2\left(\upsilon+y^2\right)^{3/2}}\right)\left[\frac{1}{2}\left(1+\frac{y}{\sqrt{\upsilon+y^2}}\right)\right]^{\varphi(j+k+1)-1} \tag{11}$$

If φ and τ are integers the index i stops at $(\tau-1)$ and k stops at $-(j+2)$. When φ is non-integer, a more general form is expressed as

$$f(y)=\sum_{i,j,k=0}^{\infty}\eta_{i,j,k}\left(\frac{1}{2\left(\upsilon+y^2\right)^{3/2}}\right)\left[1-\left(1-\left\{\frac{1}{2}\left(1+\frac{y}{\sqrt{\upsilon+y^2}}\right)\right\}\right)\right]^{\varphi(j+k+1)-1} \tag{12}$$

where $\eta_{i,j,k}=\varphi\tau\upsilon\frac{(-1)^{i+j+k}(i+1)^j}{j!}\binom{\tau-1}{i}\binom{-(j+2)}{k}$

Using the generalized binomial series, the GOGE$_{ST}$ density function $f(y)$ is given as

$$f(y)=w_{i,j,k,l,h}\,y^h\left(\upsilon+y^2\right)^{-\left(\frac{h+3}{2}\right)} \tag{13}$$

where $w_{i,j,k,l,h}=\sum_{i,j,k,l,h=0}^{\infty}\frac{(-1)^{l+h}}{2^{l+1}}\eta_{i,j,k}\binom{\varphi(j+k+1)-1}{l}\binom{l}{h}$

Secondly, the series expansion form of the GOGE$_{ST}$ distribution function $F(y)$ is expressed as

$$F(y)=\psi_{i,j,k,l,m}\,y^m\left(\upsilon+y^2\right)^{-\frac{m}{2}} \tag{14}$$

where $w_{i,j,k,l,m}=\sum_{i,j,k,l,m=0}^{\infty}\frac{(-1)^{i+j+k+l+m}}{j!2^l}\binom{\tau}{i}\binom{j}{k}\binom{\varphi(j+k)}{l}\binom{l}{m}$

3.3 Moments

The expression for the gth moment about the origin of the GOGE$_{ST}$ distribution is derived. Let $Y \sim GOGE_{ST}(\varphi,\tau,\upsilon)$ be a random variable, the gth moment of Y about the origin is defined as:

$$\mu'_g=\mathrm{E}\left(y^g\right)=\int_{-\infty}^{+\infty}y^r f(y)\,dy \tag{15}$$

Taboga (2017) showed that the gth moment in Equation (15) can be expressed as

$$\mu'_g=\mathrm{E}\left(y^g\right)=\left(1+(-1)^g\right)\int_0^{+\infty}y^g f(y)\,dy \tag{16}$$

In this context, $f(y)$ is the series expansion of the PDF in Equation (13). Hence, the g[th] moment expression of the GOGE$_{ST}$ distribution is given as:

$$\mu_g' = \left(1+(-1)^g\right)\int_0^{+\infty} y^g w_{i,j,k,l,h} y^h \left(\upsilon + y^2\right)^{-\left(h+3/2\right)} dy \qquad (17)$$

Therefore, the gth moment of the GOGE$_{ST}$ model is well-defined only when $r = 2,4,6,8,...$ (even numbers) as

$$\mu_g' = w_{i,j,k,l,h}\upsilon^{\frac{g-2}{2}} \mathrm{B}\left(\frac{g+h+1}{2},\frac{2-g}{2}\right) \qquad (18)$$

where $w_{i,j,k,l,h} = \sum_{i,j,k,l,h=0}^{\infty} \frac{(-1)^{l+h}}{2^{l+1}} \eta_{i,j,k} \binom{\varphi(j+k+1)-1}{l}\binom{l}{h}$

Corollary 1: The incomplete moment of the GOGE$_{ST}$ distribution is investigated. This suffices that the g[th] incomplete moment of a random variable Y is a function $\varphi_r'(t)$ defined as

$$\varphi_g'(t) = \mathrm{E}\left(Y^g \middle| Y < t\right) = \int_0^t y^g f(y)\, dy$$

In this context, $f(y)$ is the series expansion of the pdf in Equation (13). Hence, the g[th] incomplete moment of the GOGE$_{ST}$ distribution is expressed as

$$\varphi_r'(t) = \int_0^t y^r w_{i,j,k,l,h} y^h \left(\upsilon + x^2\right)^{-\left(h+3/2\right)} dy \qquad (19)$$

Simplifying Equation (19), the g[th] incomplete moment of the GOGE$_{ST}$ distribution is given as

$$\varphi_g'(t) = \frac{w_{i,j,k,l,h}}{2}\upsilon^{\frac{g-2}{2}} \mathrm{B}\left(t, \frac{g+h+1}{2},\frac{2-g}{2}\right) \qquad (20)$$

where $w_{i,j,k,l,h} = \sum_{i,j,k,l,h=0}^{\infty} \frac{(-1)^{l+h}}{2^{l+1}} \eta_{i,j,k} \binom{\varphi(j+k+1)-1}{l}\binom{l}{h}$

Remark: The first incomplete moment $\varphi_1'(t) = \int_0^t y f(y)\, dy$ of the GOGE$_{ST}$ distribution can be obtained by inserting $g=1$ in Equation (20).

Corollary 2: The characteristics function of the GOGE$_{ST}$ distribution is investigated. This suffices that the characteristics function of a random variable Y is a function $\Phi_Y(t)$ expressed as

$$\Phi_Y(t) = \mathrm{E}\left(e^{itY}\right) = \int_{-\infty}^{+\infty} e^{ity} f(y)\, dy = \sum_{g=0}^{\infty} \frac{(it)^g}{g!}\mu_g' \qquad (21)$$

In this context, μ_g' is the g[th] moment of the GOGE$_{ST}$ distribution in Equation (18). The characteristics function of the GOGE$_{ST}$ distribution is given as

$$\Phi_Y(t) = \sum_{g=0}^{\infty} \frac{(it)^g}{g!} w_{i,j,k,l,h}\upsilon^{\frac{g-2}{2}} \mathrm{B}\left(\frac{g+h+1}{2},\frac{2-g}{2}\right) \qquad (22)$$

where $w_{i,j,k,l,h} = \sum_{i,j,k,l,h=0}^{\infty} \frac{(-1)^{l+h}}{2^{l+1}} \eta_{i,j,k} \binom{\varphi(j+k+1)-1}{l}\binom{l}{h}$

3.4 The Entropies

The variation of uncertainty in a random variable is normally measured by the entropy. The Rényi entropy is expressed as

$$\mathrm{I}_\delta(Y) = \frac{1}{1-\delta}\log\int_{-\infty}^{+\infty} f(y)^\delta\, dy, \qquad \delta > 0 \text{ and } \delta \neq 1 \qquad (23)$$

Using the PDF (5), $f(y)^\delta$ is expressed in the series expansion form as:

$$f(y)^\delta = w_{i,j,k,l,h} y^h \left(\upsilon + y^2\right)^{-\left(h+3\delta/2\right)} \qquad (24)$$

where

$$w_{i,j,k,l,h} = (\varphi\tau\upsilon)^\delta \sum_{i,j,k,l,h=0}^{\infty} \frac{(-1)^{i+j+k+l+h}(i+\delta)^j}{j!\,2^{l+\delta}} \binom{\delta(\tau-1)}{i}\binom{-(j+2\delta)}{k}\binom{\varphi(j+k+\delta)-\delta}{l}\binom{l}{h}$$

Hence, the Rényi entropy of the GOGE$_{ST}$ distribution is given as:

$$\mathrm{I}_\delta(Y) = \frac{1}{1-\delta}\log\int_{-\infty}^{+\infty} w_{i,j,k,l,h} y^h \left(\upsilon + y^2\right)^{-\left(h+3\delta/2\right)} dy \qquad (25)$$

Simplifying and solving Equation (25) based on Taboga (2017), the Rényi entropy of the GOGE$_{ST}$ distribution is given as:

$$\mathrm{I}_\delta(X) = \frac{1}{1-\delta}\log\left(w_{i,j,k,l,h}\upsilon^{\frac{1-3\delta}{2}} \mathrm{B}\left(\frac{h+1}{2},\frac{3\delta-1}{2}\right)\right) \qquad (26)$$

Similarly, the Tsallis q-entropy is expressed as

$$\mathrm{H}_q(Y)=\frac{1}{q-1}\left(1-\left[\int_{-\infty}^{+\infty} f(y)^q\,dy\right]\right),\ q>0 \text{ and } q\neq 0 \tag{27}$$

Using the steps in Equations (24)-(26), the q-entropy of the GOGE$_{ST}$ distribution is given as:

$$\mathrm{H}_q(Y)=\frac{1}{q-1}\left\{1-\left(w^{*}_{i,j,k,l,h}\upsilon^{\frac{1-3q}{2}}\mathrm{B}\left(\frac{h+1}{2},\frac{3q-1}{2}\right)\right)\right\} \tag{28}$$

where

$w^{*}_{i,j,k,l,h}=$

$$(\varphi\tau\upsilon)^q\sum_{i,j,k,l,h=0}^{\infty}\frac{(-1)^{i+j+k+l+h}(i+q)^j}{j!2^{l+q}}\binom{q(\tau-1)}{i}\binom{-(j+2q)}{k}\binom{\varphi(j+k+q)-q}{l}\binom{l}{h}$$

3.5 The Order Statistics

Let $Y_1,Y_2,\ldots,Y_n$ be a random sample from a continuous distribution and $Y_{1:n}<Y_{2:n}<\ldots<Y_{n:n}$ are the order statistics obtained from the sample. The rth order statistic $Y_{r:n}$ is defined as

$$f_{r:n}(y)=\frac{g(y)}{\mathrm{B}(r,n-r+1)}\left[G(y)\right]^{r-1}\left[1-G(y)\right]^{n-r} \tag{29}$$

where $G(y)$ and $g(y)$ are the CDF and PDF of GOGE$_{ST}$ distribution, $\mathrm{B}(.,.)$ represent the beta function expression. Given that $0<G(y)<1$ for $y>0$, the expression in (29) can be rewritten as:

$$f_{r:n}(y)=\frac{1}{\mathrm{B}(r,n-r+1)}\sum_{l=0}^{n-r}(-1)^l\binom{n-r}{l}\left[G(y)\right]^{r+l-1}g(y) \tag{30}$$

Inserting (5) and (6) into (30) and applying series expansion, the rth order statistics for the GOGE$_{ST}$ distribution is given as

$$f_{r:n}(y)=\frac{1}{\mathrm{B}(r,n-r+1)}\vartheta_{l,i,j,k,g,h}\,y^h\left(\upsilon+y^2\right)^{-\left(\frac{h+3}{2}\right)} \tag{31}$$

where $\vartheta_{l,i,j,k,g,h}=\varphi\tau\upsilon\sum_{l=0}^{n-r}\sum_{i,j,k,g,h=0}^{\infty}\frac{(-1)^{i+j+kl+g+h}(i+1)^j}{j!2^{g+1}}\binom{\tau(r+l)-1}{i}\binom{-(j+2)}{k}\binom{n-r}{l}\binom{\varphi(j+k+1)-1}{g}\binom{g}{h}$

Remark: The minimum and maximum order statistics is obtained by setting $r=1$ and $r=n$ in (31).

IV. The Estimation and Simulation Study of the GOGE$_{ST}$ Distribution

4.1 Maximum Likelihood Estimation (MLE)

Let $y_1,y_2,\ldots,y_n$ be the observed random values from the $GOGE_{ST}(y;\varphi,\tau,\upsilon)$ distribution. Using Equation (6), the log-likelihood function of GOGE$_{ST}$ is given by

$$\begin{aligned} l(\varphi,\tau,\upsilon)&=n\log\varphi+n\log\tau+n\log\upsilon-n\log 2-\frac{3}{2}\sum_{i=1}^{n}\log\left(\upsilon+y_i^2\right)+(\varphi-1)\sum_{i=1}^{n}\log w\\ &-\sum_{i=1}^{n}\left\{\frac{w^{\varphi}}{1-w^{\varphi}}\right\}+(\tau-1)\sum_{i=1}^{n}\log\left\{1-e^{\frac{-w^{\varphi}}{1-w^{\varphi}}}\right\}-2\sum_{i=1}^{n}\log\left(1-w^{\varphi}\right)\end{aligned} \tag{32}$$

where $w=\left(\frac{1}{2}\left(1+\frac{y_i}{\sqrt{\upsilon+y_i^2}}\right)\right)$. The partial derivatives with respect to the parameters, the normal equations are derived from Equation (32) as follows:

$$\frac{\partial l}{\partial \varphi}=\frac{n}{\varphi}+\sum_{i=1}^{n}\log w-\sum_{i=1}^{n}\frac{w^{\varphi}\log w}{\left\{1-w^{\varphi}\right\}^{2}}-(\tau-1)\sum_{i=1}^{n}\frac{w^{\varphi}\log w}{\left\{1-w^{\varphi}\right\}^{2}\left\{1-e^{\frac{-w^{\varphi}}{1-w^{\varphi}}}\right\}}e^{\frac{-w^{\varphi}}{1-w\varphi}}+2\sum_{i=1}^{n}\frac{w^{\varphi}\log w}{\left\{1-w^{\varphi}\right\}}$$

$$\frac{\partial l}{\partial \tau}=\frac{n}{\tau}+\sum_{i=1}^{n}\log\left\{1-e^{\frac{-w^{\varphi}}{1-w^{\varphi}}}\right\}$$

$$\frac{\partial l}{\partial \upsilon}=\frac{n}{\upsilon}-\frac{3}{2}\sum_{i=1}^{n}\frac{1}{\left(\upsilon+y_i^{2}\right)}+(\varphi-1)\sum_{i=1}^{n}\frac{y_i}{4\left(\upsilon+y_i^{2}\right)^{3/2}w}-\varphi\sum_{i=1}^{n}\frac{y_i w^{\varphi-1}}{4\left(\upsilon+y_i^{2}\right)^{3/2}\left(1-w^{\varphi}\right)^{2}}$$
$$-\varphi(\tau-1)\sum_{i=1}^{n}\frac{y_i\left(\upsilon+y_i^{2}\right)^{\varphi-1}}{4\left(\upsilon+y_i^{2}\right)^{3/2}\left(1-w^{\varphi}\right)^{2}\left(1-e^{\frac{-w^{\varphi}}{1-w^{\varphi}}}\right)}-\varphi\sum_{i=1}^{n}\frac{y_i w^{\varphi-1}}{2\left(\upsilon+y_i^{2}\right)^{3/2}\left(1-w^{\varphi}\right)}$$

The ML estimates $\langle\hat{\varphi},\hat{\tau},\hat{\upsilon}\rangle$ of the parameters $\langle\varphi,\tau,\upsilon\rangle$, are the simultaneous solutions of the functions: $\partial l/\partial\varphi=0$, $\partial l/\partial\tau=0$ and $\partial l/\partial\upsilon=0$. The nonlinear functions, i.e., the ML estimates cannot be obtained in explicit forms but solved numerically via iterative methods. Mathematica, MATLAB, R (optim function), Maple or SAS can be utilized in finding the parameter estimates.

The performance of the $GOGE_{ST}$ parameter estimates is examined using simulation study. 10,000 samples of sizes $n=50,300$ and 1000 is generated from the $GOGE_{ST}$ distribution using R-software. The precision of the ML estimates is evaluated by using the following performance measures: mean estimates (MEs), absolute bias (Absbias), mean square errors (MSEs) and root mean square roots (RMSEs).

4.2 Simulation Study

Table 2: Numerical values of the MEs, Absbias, MSEs and RMSEs of Parameters Estimates

		$\varphi=1.0$ $\tau=1.5$ $\upsilon=1.3$						$\varphi=1.2$ $\tau=1.7$ $\upsilon=1.5$			
n	Par.	ME	Absbias	MSE	RMSE	n	Par.	ME	Absbias	MSE	RMSE
50	φ	0.9867	0.0133	0.0600	0.2449	50	φ	1.1627	0.0373	0.0489	0.2211
	τ	1.7725	0.2725	0.5807	0.7620		τ	1.9369	0.2369	0.3431	0.5857
	υ	1.4004	0.1004	0.2639	0.5137		υ	1.5823	0.0823	0.1796	0.4238
300	φ	0.9954	0.0046	0.0234	0.1529	300	φ	1.1760	0.0240	0.0220	0.1485
	τ	1.6016	0.1016	0.1836	0.4285		τ	1.8159	0.1159	0.1422	0.3770
	υ	1.3476	1.3476	0.0798	0.2824		υ	1.5585	0.0585	0.0622	0.2495
1000	φ	0.9968	0.0032	0.0100	0.0998	1000	φ	1.1815	0.2388	0.0119	0.1091
	τ	1.5450	0.0450	0.0688	0.2623		τ	1.7726	0.1109	0.0706	0.2657
	υ	1.3238	0.0238	0.0308	0.1756		υ	1.5410	0.1314	0.0295	0.1716

		$\varphi=2.2$ $\tau=2.0$ $\upsilon=2.5$						$\varphi=3.5$ $\tau=3.2$ $\upsilon=3.0$			
n	Par.	ME	Absbias	MSE	RMSE	n	Par.	ME	Absbias	MSE	RMSE
50	φ	2.3705	0.1705	0.4590	0.6775	50	φ	3.8316	0.3316	1.0869	1.0425
	τ	2.0869	0.0869	0.3903	0.6247		τ	3.3826	0.1826	0.8282	0.9101
	υ	2.4259	0.0741	0.4094	0.6399		υ	2.8747	0.1253	0.7066	0.8406
300	φ	2.2162	0.0162	0.1382	0.3717	300	φ	3.5214	0.0214	0.3828	0.6187
	τ	2.0700	0.0700	0.1627	0.4034		τ	3.3147	0.1147	0.3162	0.5623
	υ	2.5358	0.0358	0.1618	0.4022		υ	3.0557	0.0557	0.2872	0.5359
1000	φ	2.1917	0.0083	0.0748	0.2734	1000	φ	3.4751	0.0249	0.2108	0.4591
	τ	2.0547	0.0547	0.0923	0.3037		τ	3.2830	0.0830	0.1657	0.4070
	υ	2.5422	0.0422	0.0898	0.2997		υ	3.0647	0.0647	0.1587	0.3983

The results in Table 2, indicates that the parameter estimates are quite stable and very close to the true parameter values for the various sample sizes. The mean estimates tend to be closer to the true parameter values, and the absbias, MSEs and RMSEs decreases as the sample size increases. Hence, the MLE is suitable for estimating the $GOGE_{ST}$ parameters based on the simulation study.

V. The Asymmetric GARCH-Type Models

Volatility modeling of financial time series is a vital area for both consultants and economic expert. The autoregressive conditional heteroscedasticity (ARCH) model was developed for modeling time-varying volatility (Engle, 1982; Bollerslev, 1986). Asymmetric GARCH-type models such as the exponential GARCH (EGARCH) model (Nelson, 1991), Nonlinear Asymmetric GARCH (AGARCH) model (Engle and Ng, 1993), threshold GARCH (TGARCH) model by (Zakoian, 1994), asymmetric power ARCH (APARCH) model (Ding *et al.,* 1993), Absolute Value GARCH (AVGARCH) model (Taylor, 1986; Schwert, 1990), Quadratic GARCH (QGARCH) model (Sentana, 1995) and Glosten-Jagannathan-Runkle GARCH (GJRGARCH) model Glosten *et al.* (1993) were created to allow for leverage effects in the conditional variance. The daily asset log-return is denoted as r_t, and the GARCH (1,1) model is expressed as follows:

$$\begin{aligned} r_t &= \mu + \varepsilon_t, \\ \varepsilon_t &= z_t\sqrt{h_t^2}, \quad z_t \sim i.i.d. \\ h_t^2 &= \omega + \eta_1\varepsilon_{t-1}^2 + \eta_2 h_{t-1}^2, \end{aligned} \tag{33}$$

where $\omega > 0$, $\eta_1 > 0$, $\eta_2 > 0$, z_t is the conditional innovation distribution with $\mathrm{E}(z_t) = 0$ and $\mathrm{var}(z_t) = 1$. The conditional variance and mean are denoted as h_t^2 and μ_t, respectively. In this research, the EGARCH, TGARCH and APARCH models are considered. The EGARCH (1,1) model conditional variance is expressed as:

$$\ln h_t^2 = \omega + \eta_1 \frac{\varepsilon_{t-1}}{\sqrt{h_{t-1}^2}} + \eta_3 \left| \frac{\varepsilon_{t-1}}{\sqrt{h_{t-1}^2}} \right| + \eta_2 \ln h_{t-1}^2 \quad (34)$$

where η_3 is the leverage effect parameter, $In(h_t^2)$, $In(h_{t-1}^2)$ are the conditional log variance for present and preceding days, respectively, and ε_{t-1} is the accompanying error.
The TGARCH (1,1) model conditional variance is expressed as:

$$h_t = \omega + \eta_1\varepsilon_{t-1}^2 + \eta_3 I_{t-1}\varepsilon_{t-1}^2 + \eta_2 h_{t-1} \tag{35}$$

where η_3 is the leverage effect, and I_{t-1} is an indicator function expressed as:

$$I_{t-1} = \begin{cases} 1, & if \quad \varepsilon_{t-1} < 0, \\ 0, & if \quad \varepsilon_{t-1} \ge 0, \end{cases}$$

The conditional variance specification of the APARCH (1,1) is expressed as

$$h_t^\delta = \omega + \eta_1\left(|\varepsilon_{t-1}| - \eta_3\varepsilon_{t-1}\right)^\delta + \eta_2 h_{t-1}^\delta \tag{36}$$

where δ is the Taylor (power effect) parameter for the Box-Cox transformation. However, the APARCH (1,1) model converges to the TGARCH (1,1) model when $\delta = 1$. In the asymmetric GARCH-type models, positive shocks have a smaller effect on volatility than negative shocks when the parameter η_3 is positive. Harvey and Chakravarty (2008) noted that the assumption of the distribution on the innovations of the GARCH volatility models directly impacts on the accuracy of volatility estimates. Hence, the rest of this section is dedicated to the GARCH models with standardized skew normal and heavy-tailed distributions.

5.1 The Skew Normal Distribution

For the skew normal distributed innovations, the density function is given by

$$f(z) = \frac{1}{\kappa\pi} e^{\frac{-(z-\xi)^2}{2\kappa^2}} \int_{-\infty}^{\alpha\frac{z-\xi}{\kappa}} e^{\frac{-t^2}{2}} \partial t,$$

where α is the skew parameter and (ξ, κ) are the location and scale parameters, respectively.

5.2 The Skew Student-t Distribution

For the skew Student-t distributed innovations, the density function is given by

$$f(z;v) = \frac{\Gamma\left(\frac{v+1}{2}\right)\left(\frac{2}{\xi + 1/\xi}\right)}{\sqrt{\pi(v-2)}\,\Gamma\left(\frac{v}{2}\right)} \frac{s}{\left[1 + \frac{(sz+m)^2 \xi^{-2I_t}}{v-2}\right]^{\left(\frac{1+v}{2}\right)}}$$

where v is the degree of freedom, $\Gamma(.)$ denote the gamma function, ξ is the asymmetry parameter, and

$$s=\sqrt{\left(\xi^2+\frac{1}{\xi^2}-1\right)-m^2},\quad m=\frac{\Gamma\left(\frac{v+1}{2}\right)\sqrt{v-2}}{\sqrt{\pi}\,\Gamma\left(\frac{v}{2}\right)}\left(\xi-\frac{1}{\xi}\right),\quad I_t=\begin{cases}1 & if\quad z_t\geq-\frac{m}{s}\\ -1 & if\quad z_t<-\frac{m}{s}\end{cases}$$

.

5.3 The Skew Generalized Error Distribution

For the generalized error distributed innovations, the density function is given by

$$f\left(z\,|\,v,\eta\right)=C\exp\left(-\frac{1}{\left[1-sign\left(z-\kappa\right)\eta\right]^{v}\varphi^{v}}\left|z-\kappa\right|^{v}\right)$$

where v is the degrees of freedom, η is the skew parameter, $(-1<\eta<1)$, $C=v\left[2\varphi\Gamma\left(v^{-1}\right)\right]^{-1}$,

$\kappa=2\eta AS\left(\eta\right)^{-1}$,

$A=\Gamma\left(2/v\right)\Gamma\left(1/v\right)^{-\frac{1}{2}}\Gamma\left(3/v\right)^{-\frac{1}{2}}$,

$\varphi=\Gamma\left(1/v\right)^{\frac{1}{2}}\Gamma\left(3/v\right)^{-\frac{1}{2}}S\left(\eta\right)^{-1}$,and

$S\left(\eta\right)=\sqrt{1+3\eta^2-4A^2\eta^2}$.

5.4 Generalized Hyperbolic Distribution

For the generalized hyperbolic distributed innovations, the density function is given by

$$f\left(z;\lambda,\alpha,\beta,\delta,\mu\right)=\frac{\left\{\alpha^2-\beta^2\right\}^{\lambda/2}}{\sqrt{2\pi}\,\alpha^{\lambda-\frac{1}{2}}\delta^{\lambda}\Psi_{\lambda}\left\langle\delta\sqrt{\alpha^2-\beta^2}\right\rangle}\left(\delta^2+\left\langle z-\mu\right\rangle^2\right)^{\left(\lambda-\frac{1}{2}\right)/2}$$
$$\times\Psi_{\lambda-\frac{1}{2}}\left\{\alpha\sqrt{\delta^2+\left\langle z-\mu\right\rangle^2}\right\}\exp\left(\beta\left\{z-\mu\right\}\right)$$

where δ is scale parameter, μ is location parameter, β is the asymmetry parameter, λ,α are real parameters, Ψ_{λ} is the modified Bessel function.

5.5 Johnson Reparametrized (SU) Distribution

For the Johnson reparametrized (SU) distributed innovations, the density function is given by

$$f\left(z;\eta,\tau,\upsilon,\vartheta\right)=\frac{\vartheta}{\eta\sqrt{1+\left(\frac{z-\tau}{\eta}\right)^2}}\phi\left[\upsilon+\vartheta\sinh^{-1}\left(\frac{z-\tau}{\eta}\right)\right]$$

where, ϕ is the density function of $N\left(0,1\right)$, τ,η are location and scale parameters, respectively, while υ,ϑ denote the skew and kurtosis parameters respectively.

5.6 The Standardized Generalized Odd Generalized Exponentiated Skew-t Distribution

The standardized GOGE$_{ST}$ density is derived by introducing arbitrary location and scale parameters μ and σ via the transformation $z=\frac{y-\mu}{\sigma}$. However, Engle (1982) defined the errors as an autoregressive conditional heteroscedastic process, where error terms (ε_t) is expressed as $\varepsilon_t=z_t\sqrt{h_t^2}$, where $\mathrm{E}(z_t)=0$ and $\mathrm{var}(z_t)=1$. The random variable z_t can be expressed as follows: $z_t=\varepsilon_t\big/\sqrt{h_t^2}$ and $\partial z_t/\partial\varepsilon_t=1\big/\sqrt{h_t^2}$. Hence, the standardized density function of the GOGE$_{ST}$ is given by

$$f(\varepsilon_t;\varphi,\tau,\upsilon,\sqrt{h_t^2})=\left(\frac{1}{\left\{h_t^2\right\}^{1/2}}\right)\frac{\varphi\tau\upsilon\left[\frac{1}{2}\left(1+\frac{\frac{\varepsilon_t}{\sqrt{h_t^2}}}{\sqrt{\upsilon+\left(\frac{\varepsilon_t}{\sqrt{h_t^2}}\right)^2}}\right)\right]^{\varphi-1}e^{\frac{-\left(\frac{1}{2}\left(1+\frac{\frac{\varepsilon_t}{\sqrt{h_t^2}}}{\sqrt{\upsilon+\left(\frac{\varepsilon_t}{\sqrt{h_t^2}}\right)^2}}\right)\right)^{\varphi}}{1-\left(\frac{1}{2}\left(1+\frac{\frac{\varepsilon_t}{\sqrt{h_t^2}}}{\sqrt{\upsilon+\left(\frac{\varepsilon_t}{\sqrt{h_t^2}}\right)^2}}\right)\right)^{\varphi}}}}{2\left(\upsilon+\left(\frac{\varepsilon_t}{\sqrt{h_t^2}}\right)^2\right)^{3/2}\left[1-\left(\frac{1}{2}\left(1+\frac{\frac{\varepsilon_t}{\sqrt{h_t^2}}}{\sqrt{\upsilon+\left(\frac{\varepsilon_t}{\sqrt{h_t^2}}\right)^2}}\right)\right)^{\varphi}\right]^2}\left\{1-e^{\frac{-\left(\frac{1}{2}\left(1+\frac{\frac{\varepsilon_t}{\sqrt{h_t^2}}}{\sqrt{\upsilon+\left(\frac{\varepsilon_t}{\sqrt{h_t^2}}\right)^2}}\right)\right)^{\varphi}}{1-\left(\frac{1}{2}\left(1+\frac{\frac{\varepsilon_t}{\sqrt{h_t^2}}}{\sqrt{\upsilon+\left(\frac{\varepsilon_t}{\sqrt{h_t^2}}\right)^2}}\right)\right)^{\varphi}}}\right\}^{\tau-1} \quad (37)$$

The log-likelihood function (LL) is given by

$$
\begin{aligned}
LL(\zeta) = {} & n\log\varphi + n\log\tau + n\log\upsilon - n\log 2 - \frac{3}{2}\sum_{t=1}^{n}\log\left(\upsilon+\left(\frac{\varepsilon_t}{\sqrt{h_t^2}}\right)^2\right) + (\varphi-1)\sum_{t=1}^{n}\log\left(\frac{1}{2}\left(1+\frac{\varepsilon_t}{\sqrt{h_t^2}\sqrt{\upsilon+\left(\frac{\varepsilon_t}{\sqrt{h_t^2}}\right)^2}}\right)\right) \\
& -2\sum_{t=1}^{n}\log\left\{1-\left(\frac{1}{2}\left(1+\frac{\varepsilon_t}{\sqrt{h_t^2}\sqrt{\upsilon+\left(\frac{\varepsilon_t}{\sqrt{h_t^2}}\right)^2}}\right)\right)^{\varphi}\right\} - \sum_{t=1}^{n}\left\{\frac{\left(\frac{1}{2}\left(1+\frac{\varepsilon_t}{\sqrt{h_t^2}\sqrt{\lambda+\left(\frac{\varepsilon_t}{\sqrt{h_t^2}}\right)^2}}\right)\right)^{\alpha}}{1-\left(\frac{1}{2}\left(1+\frac{\varepsilon_t}{\sqrt{h_t^2}\sqrt{\upsilon+\left(\frac{\varepsilon_t}{\sqrt{h_t^2}}\right)^2}}\right)\right)^{\alpha}}\right\} \\
& +(\tau-1)\sum_{t=1}^{n}\log\left\{1-e^{\frac{-\left(\frac{1}{2}\left(1+\frac{\varepsilon_t}{\sqrt{h_t^2}\sqrt{\upsilon+\left(\frac{\varepsilon_t}{\sqrt{h_t^2}}\right)^2}}\right)\right)^{\varphi}}{1-\left(\frac{1}{2}\left(1+\frac{\varepsilon_t}{\sqrt{h_t^2}\sqrt{\upsilon+\left(\frac{\varepsilon_t}{\sqrt{h_t^2}}\right)^2}}\right)\right)^{\varphi}}}\right\} - \frac{1}{2}n\log h_t^2
\end{aligned}
\tag{38}
$$

where $\zeta = \left(\varphi,\tau,\upsilon,h_t^2\right)$ and h_t^2 denote the asymmetric GARCH-type volatility models with unknown vector parameters.

VI. Evaluation of Volatility Predictions

6.1 Model Selection Criteria

The information criteria are used for the selection of the models introduced in Section 3. The accuracy of the models is appraised by the modified information criteria of Brooks and Burke (2003). The modified AIC and BIC criteria are given by

$$AIC = \frac{2c}{T} - \frac{2LL}{T}$$

$$BIC = \frac{c\log_e(T)}{T} - \frac{2LL}{T}$$

where, T is the number of observations, c is the total number of estimated parameters, and LL denotes the estimated log-likelihood value. The model with the least AIC and BIC values is regarded as the best performing model in terms of the specified conditional innovation distribution.

6.2 Forecasts Evaluation

The prediction performance of the asymmetric GARCH-type models is evaluated using the mean square error (MSE), root mean square root (RMSE), and mean absolute error (MAE). The MSE, RMSE, and MAE for the volatility forecasts are given by

$$MSE = \frac{1}{T}\sum_{t=1}^{T}\left(\hat{h}_t - h_t\right)^2$$

$$RMSE = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\hat{h}_t - h_t\right)^2}$$

$$MAE = \frac{1}{T}\sum_{t=1}^{T}\left|\hat{h}_t - h_t\right|$$

where $\hat{h}_t$ and h_t represent the volatility forecast and realized volatility, and T is the sample size. The model with the least MSE, RMSE, and MAE values is regarded as the most suitable for forecasting the volatility of the daily returns.

VII. Empirical Finding

7.1 Data report

To appraise the performance of the asymmetric GARCH-type volatility models in predicting daily volatility, Bitcoin (BTC) cryptocurrency price index is utilized. The utilized data consist 3402 daily log-returns from 02/02/2012 to 31/05/2021. The estimation process is carried out using 3251 daily log-returns for nine years (2012-2020) while the remaining 151 daily log-returns in the year 2021 are used for assessment of out-of-sample forecasting performance. Table 3 presents the summary statistics of the log-returns of the Bitcoin (BTC) and the graphical plots are depicted in Figure 2. Negative skewness and excess kurtosis are quite evident, leading to large Jarque-Bera (JB) test statistic value ($p < 0.001$) indicating that the daily log-returns are non-normally distributed. More so, the Lagrange-multiplier (LM) test specifies the incidence of ARCH effects in the conditional variance.

Table 3: Summary statistics for the Bitcoin (BTC).

	BTC
Number of observations	3251
Mean	0.2604
Median	0.1534
Minimum	-48.090
Maximum	30.830
Standard Deviation	4.4438
Skewness	-0.8566
Kurtosis	13.3007
Jarque-Bera	24398 (p < 0.001)
ARCH (2)	314.71 p < 0.001)

7.2 Estimation

The asymmetric models are estimated under the skew normal (SNORM), skew student-t (SST), skew generalized error (SGE), generalized hyperbolic (GHYB), Johnson (SU) reparametrized (JSU) and $GOGE_{ST}$ innovation densities. To obtain the parameter estimates of the models under SNORM, SST, SGE, GHYB, JSU densities, the *rugarch* package in R-software is used. The *Optim* function in R-software is utilized for the parameter estimation of the models under the $GOGE_{ST}$ innovation density. Table 4 presents the parameter estimates for the EGARCH (1,1), TGARCH (1,1) and APGARCH (1,1) models under six innovation densities. The parameter estimates of the conditional variance are highly statistically significant, and the parameter η_3 is significant at standard level which shows that the daily log-returns have leverage effect. Hence, the impact of the shocks is asymmetric in nature which implies that the impact of negative shocks on volatility are higher than positive shocks of the same size.

Tables 5 gives some important statistics for model selection, which shows that the EGARCH-$GOGE_{ST}$ has the highest log-likelihood, and least AIC and BIC values relative to other models. Hence, the EGARCH-$GOGE_{ST}$ model is selected as the best asymmetric model for the BTC log-returns conditional variance.

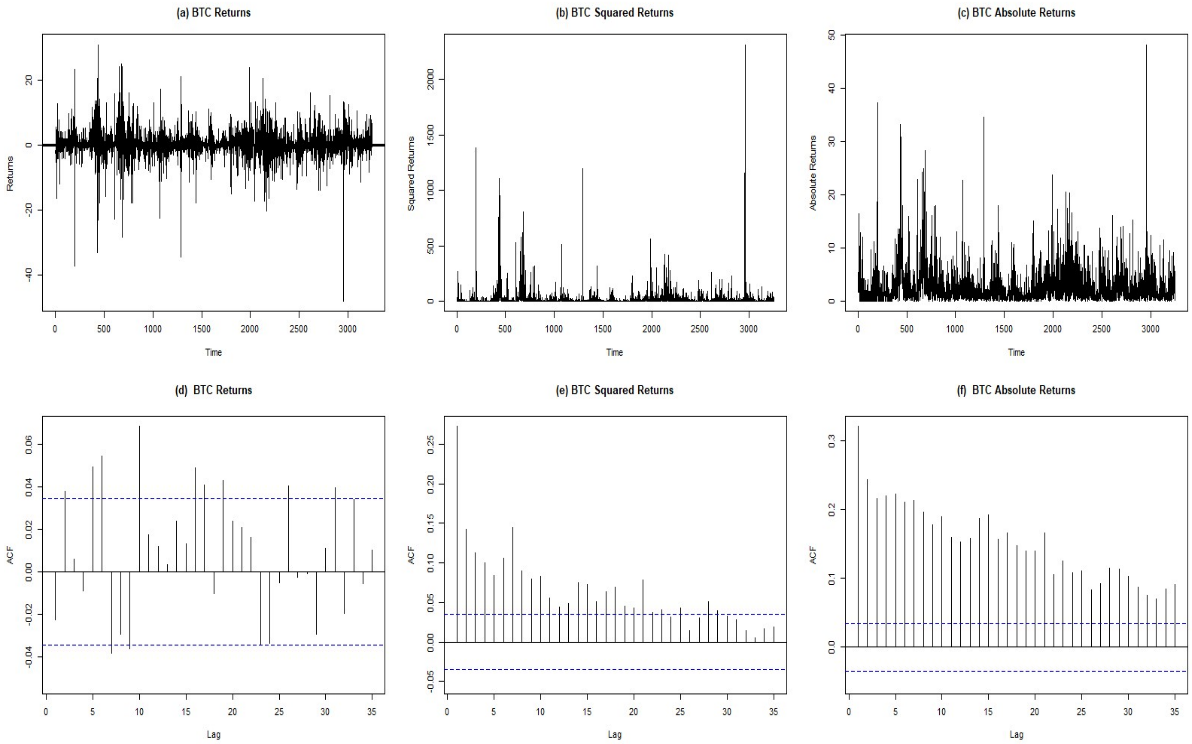

Figure 2: Bitcoin (BTC) daily log-returns, squared and absolute log-returns and sample autocorrelations.

There is no clear noticeable pattern in the BTC daily log-returns. However, there are some persistence in the squared and absolute log-returns as depicted in Figure $2_{(b,\ c)}$. Specifically, the plots show evidence of volatility clustering; that is low volatility values followed by low volatility values and high volatility values followed by high volatility values. More so, the daily log-returns show no clear evidence of serial correlation, but the squared and absolute log-returns are positively autocorrelated as depicted in Figures $2_{(e,\ f)}$.

Table 4: Parameter estimates of the asymmetric models for the BTC log-returns under six innovation densities.

EGARCH (1,1)						
Par.	SNORM	SSTD	SGED	GHYP	JSU	$GOGE_{ST}$
μ	0.0000	0.1671'***'	0.0000	0.1025''	0.1398'*'	4.402E-04
ω	0.2447'***'	0.0684'***'	0.0850'***'	0.0674'***'	0.0660'***'	-1.713E-04
η_1	-0.0232'.'	0.0419'*'	0.0153''	0.0362'.'	0.0359'.'	0.0332 '.'
η_2	0.9248'***'	0.9772'***'	0.9670'***'	0.9752'***'	0.9753'***'	0.8119'***'
η_3	0.2933'***'	0.3236'***'	0.2852'***'	0.2928'***'	0.2905'***'	-0.0350
α	0.8873'***'	-	-	0.2500'.'	-	-
ξ	-	0.9856'***'	-	-	-	-
ν	-	2.6900'***'	0.8226'***'	-	-	-
η	-	-	1.0000'***'	-	-	-
β	-	-	-	-0.1765	-	-
λ	-	-	-	-1.0394'***'	-	-
υ	-	-	-	-	-0.0433	-
ϑ	-	-	-	-	1.0531'***'	3.0281'***'
γ	-	-	-	-	-	7.395E+01 '***'
τ	-	-	-	-	-	0.2964

TGARCH (1,1)						
Par.	SNORM	SST	SGE	GHYB	JSU	$GOGE_{ST}$
μ	0.0075	0.1490'*'	0.0000	0.0908''	0.1272'*'	3.95E-03
ω	0.2854'***'	0.0902'*'	0.1168'***'	0.0895'*'	0.0886'***'	0.2853'**'
η_1	0.1681'***'	0.2050'***'	0.1801'***'	0.1857'***'	0.1843'***'	1.2513'***'
η_2	0.8184'***'	0.8652'***'	0.8521'***'	0.8621'***'	0.8626'***'	0.3644'***'
η_3	0.1036'.'	-0.0941'.'	-0.0929	-0.0890	-0.0883	0.0549
α	0.8881'***'	-	-	0.2500'.'	-	-
ξ	-	0.9803'***'	-	-	-	-
ν	-	2.6880'***'	0.8220'***'	-	-	-
η	-	-	1.0000'***'	-	-	-
β	-	-	-	-0.1848''	-	-
λ	-	-	-	-1.0349'***'	-	-
υ	-	-	-	-	-0.0491	-
ϑ	-	-	-	-	1.0519'***'	1.5285'***'
γ	-	-	-	-	-	1.716E+02 '***'
τ	-	-	-	-	-	1.227E+01

APARCH (1,1)						
Par.	SNORM	SST	SGE	GHYB	JSU	$GOGE_{ST}$
μ	0.0631	0.1467'***'	0.0000	0.0908''	0.1275'.'	3.31E-04
ω	0.4302'***'	0.0793'***'	0.1023'***'	0.0802'***'	0.0788'.'	0.0307'*'
η_1	0.1732'***'	0.1957'***'	0.1736'***'	0.1795'***'	0.1781'***'	0.0959'***'
η_2	0.8121'***'	0.8673'***''	0.8545'***'	0.8641'***'	0.8647'***'	0.7386'***'
η_3	0.0778'.'	-0.0994''	-0.0085	-0.0937	-0.0931	8.314E-03
δ	1.3160'***'	0.8933'***'	0.8730'***'	0.8995'***'	0.8956'***'	1.0677'***'
α	0.8921'***'	-	-	0.2500'.'	-	-
ξ	-	0.9797'***'	-	-	-	-
ν	-	2.6859'***'	0.8207'***'	-	-	-
η	-	-	1.0000'***'	-	-	-
β	-	-	-	-0.1838''	-	-
λ	-	-	-	-1.0331'***'	-	-
υ	-	-	-	-	-0.0488	-
ϑ	-	-	-	-	1.0512'***'	1.5395'***'
γ	-	-	-	-	-	1.382E+02 '***'
τ	-	-	-	-	-	3.879E+01

Significance levels: 0 '***', 0.001 '**', 0.01 '*', 0.05 '.', 0.1 '', 1

For the asymmetric models under the GOGE_{ST} density, the parameter estimates of the conditional variance are highly statistically significant, and the parameter η_3 is significant at standard level which shows that the daily log-returns have leverage effect. Hence, the impact of the shocks is asymmetric in nature which implies that the impact of negative shocks on volatility are higher than positive shocks of the same size. Tables 5 gives some important statistics for model selection, which shows that the asymmetric models under GOGE_{ST} innovation density have the highest log-likelihood, and least AIC and BIC values relative to other models. However, the EGARCH-GOGE_{ST} model is selected as the best volatility model for the BTC log-returns conditional variance given it has the least AIC and BIC values.

Table 5: Comparison of the Models for Selection

Model	Log-Likelihood	AIC	BIC
EGARCH- SNORM	-8930.696	5.4978	5.5090
EGARCH- SST	-8409.936	5.1781	5.1912
EGARCH-SGE	-8418.751	5.1835	5.1966
EGARCH-GHYB	-8405.539	5.1760	5.1909
EGARCH-JSU	-8404.530	5.1747	5.1878
EGARCH- GOGE_{ST}	-5929.735	3.6529	3.6678
TGARCH- SNORM	-8931.886	5.4985	5.5098
TGARCH- SST	-8404.229	5.1745	5.1877
TGARCH-SGE	-8413.927	5.1805	5.1936
TGARCH-GHYB	-8399.637	5.1723	5.1873
TGARCH-JSU	-8398.837	5.1712	5.1843
TGARCH- GOGE_{ST}	-7778.089	4.7900	4.8049
APARCH- SNORM	-8930.696	5.4978	5.5090
APARCH- SST	-8409.936	5.1781	5.1912
APARCH-SGE	-8418.751	5.1835	5.1966
APARCH-GHYB	-8405.539	5.1760	5.1909
APARCH-JSU	-8404.530	5.1747	5.1878
APARCH- GOGE_{ST}	-6620.921	4.0781	4.0931

Table 6 presents the diagnostic tests of the EGARCH-GOGE_{ST}, and other models. The Ljung-Box statistic indicates that the squared standardized residuals from the fitted models exhibit no sign of serial correlation. Likewise, the ARCH-LM statistic indicates that the standardized residuals from the estimated models exhibit no additional ARCH processes which implies that the conditional variance equations are specified correctly. Thus, all the models did well in describing the characterizing dynamics features the log-returns.

Table 6: Estimated Models Diagnostic Tests.

Model	**Ljung-Box Statistic**	**p-value**	**ARCH-LM Statistic**	**p-value**
EGARCH- SNORM	2.568	0.9998	1.685	0.9982
EGARCH- SST	3.138	0.9995	2.026	0.9961
EGARCH-SGE	3.249	0.9993	2.046	0.9960
EGARCH-GHYB	3.157	0.9994	2.033	0.9961
EGARCH-JSU	3.155	0.9995	2.036	0.9961
EGARCH- GOGE_{ST}	21.297	0.8784	20.062	0.9149
TGARCH- SNORM	3.429	0.9991	2.641	0.9887

TGARCH- SST	4.158	0.9972	3.176	0.9770
TGARCH-SGE	3.806	0.9983	2.749	0.9867
TGARCH-GHYB	4.107	0.9974	3.119	0.9785
TGARCH-JSU	4.114	0.9973	3.129	0.9782
TGARCH- $GOGE_{ST}$	4.849	0.9933	4.274	0.9341
APARCH- SNORM	2.948	0.9996	2.064	0.9958
APARCH- SST	4.902	0.9929	3.946	0.9497
APARCH-SGE	4.436	0.9959	3.395	0.9705
APARCH-GHYB	4.755	0.9940	3.788	0.9564
APARCH-JSU	4.798	0.9937	3.839	0.9543
APARCH- $GOGE_{ST}$	4.179	0.9971	3.499	0.9671

4.3 Forecasts

The out-of-sample performance of the asymmetric models are presented in Table 7, while Table 8 presents the comparison of the innovation densities through ranking. The evaluation measures indicates that the asymmetric models under the $GOGE_{ST}$ density have the least mean square error (MSE), Root mean square error (RMSE), and mean absolute error (MAE) values. However, it indicates that the TGARCH-$GOGE_{ST}$ model is statistically effective and displays superior ability in forecasting the BTC volatility relative to others, having the least MSE, RMSE and MAE values. More so, the $GOGE_{ST}$ innovation density has the least total rank value based on the MSE, RMSE and MAE for the EGARCH (1,1), TGARCH (1,1) and APARCH (1,1) models.

Table 7: Forecasts Evaluation of the Estimated Models.

Model	MSE	RMSE	MAE
EGARCH- SNORM	23.5393	4.8517	3.5653
EGARCH- SST	23.5110	4.8488	3.5709
EGARCH-SGE	23.5394	4.8517	3.5653
EGARCH-GHYB	23.5153	4.8493	3.5678
EGARCH-JSU	23.5118	4.8489	3.5695
EGARCH- $GOGE_{ST}$	22.8307	4.7781	3.4785
TGARCH- SNORM	23.5369	4.8515	3.5652
TGARCH- SST	23.5114	4.8488	3.5699
TGARCH-SGE	23.5393	4.8517	3.5653
TGARCH-GHYB	23.5170	4.8494	3.5673
TGARCH-JSU	23.5127	4.8490	3.5689
TGARCH- $GOGE_{ST}$	20.4212	4.5190	3.1792
APARCH- SNORM	23.5221	4.8499	3.5664
APARCH- SST	23.5115	4.8489	3.5698
APARCH-SGE	23.5393	4.8517	3.5653
APARCH-GHYB	23.5170	4.8494	3.5673
APARCH-JSU	23.5127	4.8490	3.5689
APARCH- $GOGE_{ST}$	22.7278	4.7674	3.4577

Table 8: Forecasts Evaluation – Innovation Distributions Comparison.

	EGARCH (1,1)					
	SNORM	SST	SGE	GHYB	JSU	$GOGE_{ST}$
MSE	5	2	6	4	3	1
RMSE	5.5	2	5.5	4	3	1
MAE	2.5	6	2.5	4	5	1
TOTAL	13	10	14	12	11	3
	TGARCH (1,1)					
	SNORM	SST	SGE	GHYB	JSU	$GOGE_{ST}$
MSE	5	2	6	4	3	1
RMSE	5	2	6	4	3	1
MAE	2	6	3	4	5	1
TOTAL	12	10	15	12	11	3
	APARCH (1,1)					
	SNORM	SST	SGE	GHYB	JSU	$GOGE_{ST}$
MSE	5	2	6	4	3	1
RMSE	5	2	6	4	3	1
MAE	3	6	2	4	5	1
TOTAL	13	10	14	12	11	3

Overall, the comparison between the volatility models in this research largely supports the utilization of the asymmetric GARCH model with $GOGE_{ST}$ density for estimating and forecasting the volatility of the BTC log-returns. More so, the proposed GOG_{ST} innovation density seems generally the best for the asymmetric GARCH (1,1) models considered in this research.

VIII. Conclusion

This research shows that most financial returns have non-normal features such as heavy-tails, excess kurtosis, asymmetric volatility clustering, and skewness. Financial return volatility is a significant measure in financial decisions, such as option pricing, risk management, and portfolio selection, so it is useful to create vigorous driven conditional innovation distribution for GARCH-type volatility models. In this research, the generalized odd generalized skew-t ($GOGE_{ST}$) distribution quite capable of modeling skewness, leptokurtic behaviour, and unimodal shapes is derive. Volatility modeling of the Bitcoin (BTC) cryptocurrency using three asymmetric GARCH models with $GOGE_{ST}$ innovation density relative to the skew normal, skew student-t, skew generalized error, generalized hyperbolic, Johnson (SU) reparametrized innovation densities in terms of predictive performance using three forecast performance measures. It was found that asymmetric GARCH models with $GOGE_{ST}$ innovation density are optimally the best models based on model selection criteria, and the EGARCH- $GOGE_{ST}$ innovation density is the most capable for describing the dynamic features of the returns as it mirrors the fundamental process in relations to leptokurtic innovation, serial correlation and asymmetric volatility clustering. The research findings confirm that asymmetric GARCH models improve the predictive performance. Furthermore, the results validate the superiority of the TGARCH-$GOGE_{ST}$ model in out-of-sample forecasting performance over other models for Bitcoin volatility modeling.

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