Some Inferences on Odd Generalized Exponential-Rayleigh Distribution

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***Abstract*—The Rayleigh distribution has wide range of applications in Applied Sciences; some of which are related to sea waves, harbor, coastal engineering and studies on wind-wave heights among others. In this paper, we propose a new lifetime model entitled Odd Generalized Exponential-Rayleigh Distribution and present some of its statistical properties comprising moments, moment generating function, reliability analysis and order statistics. A method of maximum likelihood was used to estimate the parameters of the new distribution.**

Keywords*-component; Rayleigh distribution; moment; moment generating function; order statistics.*

I. Introduction

Rayleigh distribution was named after Lord Rayleigh (1842-1919) a British physicist as well as mathematician also known as John William Strutt. In 1895, he discovered the inert gas argon (Ar), the research that earned him the 1904 Nobel Prize in Physics, see [8].

The distribution has wide range of applications in the field of applied sciences, especially in modeling the lifetime of an object or service time. The applicability of this distribution spans over diverse areas of human endeavor comprising sea waves, harbor, coastal and ocean engineering, heights and periods of wind waves [1]. Other areas where the applicability of the distribution arises includes researches on random complex numbers whose real and imaginary components are independently and identically distributed normal variables, health, agriculture and biology among others [2].

Despite its applicability, the distribution like several others, suffers problem of lack of flexibility due to having only one parameter. For instance, it was stated in[6]that, the well-known classical distributions such as exponential, Rayleigh, Weibull and gamma are limited in their characteristics and are unable to show wide flexibility. Because of this and several other problems; several researchers embarked on overcoming these challenges by generalizing some of these classical distributions to come up with compound distributions.

In recent time, several researchers came up with generalizations of the Rayleigh distribution, some of which are, transmuted Rayleigh distribution [4], transmuted generalized Rayleigh distribution [3], the kumaraswamy generalized Rayleigh distribution [2] and Weibull-Rayleigh distribution by [5].

A random variable X is said to have a Rayleigh distribution if its Cumulative Distribution Function (CDF) and the probability density function (pdf) are given as:

$$G(x;\sigma) = 1 - e^{-\frac{x^2}{2\sigma^2}} \quad (1)$$

and

$$g(x;\sigma) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} \quad (2)$$

respectively, where $x > 0, \sigma > 0$ is the scale parameter.

Reference [7] comes up with a new family of distributions called Odd Generalized Exponential Family of distributions. The CDF and pdf of this family are respectively given in equations (3) and (4):

$$F(x;\alpha,\lambda,\xi) = (1 - e^{-\lambda\frac{G(x;\xi)}{1-G(x;\xi)}})^{\alpha} \quad (3)$$

$$f(x;\alpha,\lambda,\xi) = \frac{\lambda\alpha g(x;\xi)}{(1-G(x;\xi))^2} e^{-\lambda\frac{G(x;\xi)}{1-G(x;\xi)}} (1 - e^{-\lambda\frac{G(x;\xi)}{1-G(x;\xi)}})^{\alpha-1},$$

$$x > 0, \alpha, \lambda > 0 \quad (4)$$

where $1 - G(x,\xi)$ is the survival function, $G(x,\xi)$ and $g(x,\xi)$ are the CDF and pdf of the baseline distribution.

In this article, we present a new distribution which has its root from the Generalized Exponential distribution and Rayleigh distribution called the Odd Generalized

Exponential-Rayleigh (OGE-R) distribution using the family proposed by [8].

II. RESEARCH METHODOLOGY

A. Model's Development

In this section we define new three-parameter distribution called Odd Generalized Exponential-Rayleigh (OGE-R) distribution with parameters α, λ, and σ written as OGE-R($\boldsymbol{\Theta}$), where the vector $\boldsymbol{\Theta}$ is defined by $\boldsymbol{\Theta} = (\alpha, \lambda, \sigma)$.

A random variable X is said to have OGE-R distribution with parameters α, λ, and σ if its CDF given by:

$$F(x;\Theta) = (1 - e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)})^{\alpha} \qquad (5)$$

The corresponding pdf of X~OGE-R is

$$f(x;\Theta) = \frac{\alpha\lambda x}{\sigma^2} e^{\frac{x^2}{2\sigma^2}} e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)} (1 - e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)})^{\alpha-1} \qquad (6)$$

where $\alpha > 0$, $\lambda > 0$ and $\sigma > 0$ are shape, rate (inverse scale) and scale parameters respectively.
Figures1 and 2 represent the graphical plots of the CDF and pdf OGE-R distribution.

III. ANALYSIS

In this section, we study some statistical properties of OGE-R distribution, comprising moments and moment generating function.

A. The moments

In this subsection, we will derive ther^{th} moments of the OGE-R distribution as an infinite series expansion.

Theorem 1. If X~OGE-R($\boldsymbol{\Theta}$) where $\boldsymbol{\Theta} = (\alpha, \lambda, \sigma)$, then the r^{th} moment of X is given by

$$u_r' = \sum_{i=0}^{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{\alpha-1}{i}\binom{j}{k}(-1)^{i+j+k}(i+1)^j \frac{\alpha\lambda^{j+1}\sigma^r 2^{\frac{r}{2}}}{j!(k-j-1)^{\frac{r+2}{2}}}\left(\frac{r}{2}\right)!$$

Proof. The r^{th} moment of the random variable X with pdf $f(x)$ is defined by

$$u_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx \qquad (7)$$

Substituting equation (6) into (7), we get

$$u_r' = E(X^r) = \int_0^{\infty} x^r \frac{\alpha\lambda x}{\sigma^2} e^{\frac{x^2}{2\sigma^2}} e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)} (1 - e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)})^{\alpha-1}dx \qquad (8)$$

Using binomial expansion for we obtain

$$(1 - e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)})^{\alpha-1} = \sum_{k=0}^{\alpha-1}\binom{\alpha-1}{i}(-1)^i e^{-\lambda i(e^{\frac{x^2}{2\sigma^2}}-1)} \qquad (9)$$

Substituting equation (9) into (8), we get

$$u_r' = \sum_{i=0}^{\alpha-1}\binom{\alpha-1}{i}(-1)^i \frac{\alpha\lambda}{\sigma^2}\int_0^{\infty} x^r e^{\frac{x^2}{2\sigma^2}} e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)} e^{-\lambda i(e^{\frac{x^2}{2\sigma^2}}-1)}dx \qquad (10)$$

Using series expansion of $e^{-\lambda(i+1)(e^{\frac{x^2}{2\sigma^2}}-1)}$, we get

$$e^{-\lambda(i+1)(e^{\frac{x^2}{2\sigma^2}}-1)} = \sum_{j=0}^{\infty}\frac{(-1)^j\lambda^j(1+i)^j(e^{\frac{x^2}{2\sigma^2}}-1)^j}{j!} \qquad (11)$$

Substituting equation (11) into (10), we get

$$u_r' = \sum_{i=0}^{\alpha-1}\sum_{j=0}^{\infty}\binom{\alpha-1}{i}(-1)^{i+j}(i+1)^j\frac{\alpha\lambda^{j+1}}{j!\sigma^2}\int_0^{\infty} x^{r+1}e^{\frac{x^2}{2\sigma^2}}(e^{\frac{x^2}{2\sigma^2}}-1)^j dx \qquad (12)$$

Using the binomial expansion for $(e^{\frac{x^2}{2\sigma^2}}-1)^j$ we obtain

$$(e^{\frac{x^2}{2\sigma^2}}-1)^j = \sum_{k=0}^{j}\binom{j}{k}(-1)^k e^{\frac{(j-k)x^2}{2\sigma^2}} \qquad (13)$$

Substituting equation (13) into (12), we get

$$u_r' = \sum_{i=0}^{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{\alpha-1}{i}\binom{j}{k}(-1)^{i+j+k}(i+1)^j\frac{\alpha\lambda^{j+1}}{j!\sigma^2}\int_0^{\infty} x^{r+1}e^{\frac{x^2}{2\sigma^2}}e^{\frac{(j-k)x^2}{2\sigma^2}}dx \qquad (14)$$

Equation (14) can now be written as:

$$u_r' = \sum_{i=0}^{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{\alpha-1}{i}\binom{j}{k}(-1)^{i+j+k}(i+1)^j\frac{\alpha\lambda^{j+1}}{j!\sigma^2}\int_0^{\infty} x^{r+1}e^{\frac{(j-k+1)x^2}{2\sigma^2}}dx \qquad (15)$$

Equation (15) can then be rewritten as:

$$u_r' = \sum_{i=0}^{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{\alpha-1}{i}\binom{j}{k}(-1)^{i+j+k}(i+1)^j\frac{\alpha\lambda^{j+1}}{j!\sigma^2}\int_0^{\infty} x^{r+1}e^{\frac{-(k-j+1)x^2}{2\sigma^2}}dx \qquad (16)$$

Now, we let

$$y = \left(\frac{(k-j+1)x^2}{2\sigma^2}\right)^{\frac{1}{2}} \Rightarrow x = \left(\frac{2y\sigma^2}{k-j-1}\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{(k-j-1)x}{\sigma^2} \Rightarrow \quad dx = \frac{\sigma^2 dy}{(k-j-1)x}$$

Replacing x in the last expression and simplifying we have

$$dx = \frac{\sigma^2 dy}{(k-j-1)^{\frac{1}{2}}(2y\sigma^2)^{\frac{1}{2}}}$$

Substituting the above equations into (16), we get

$$u'_r = \sum_{i=0}^{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{\alpha-1}{i}\binom{j}{k}(-1)^{i+j+k}(i+1)^j \frac{\alpha\lambda^{j+1}}{j!\sigma^2}\int_0^\infty\left(\frac{2y\sigma^2}{k-j-1}\right)^{\frac{r+1}{2}} e^{-y}\frac{\sigma^2}{(k-j-1)(2y\sigma^2)}dy \quad (17)$$

After some simplification equation (17) becomes

$$u'_r = \sum_{i=0}^{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{\alpha-1}{i}\binom{j}{k}(-1)^{i+j+k}(i+1)^j \frac{\alpha\lambda^{j+1}\sigma^r 2^{\frac{r}{2}}}{j!(k-j-1)^{\frac{r+2}{2}}}\int_0^\infty y^{\frac{r}{2}}e^{-y}dy \quad (18)$$

By using the gamma function and simplifying, equation (18) becomes

$$u'_r = \sum_{i=0}^{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{\alpha-1}{i}\binom{j}{k}(-1)^{i+j+k}(i+1)^j \frac{\alpha\lambda^{j+1}\sigma^r 2^{\frac{r}{2}}}{j!(k-j-1)^{\frac{r+2}{2}}}\left(\frac{r}{2}\right)! \quad (19)$$

This completes the proof.

B. The moment generating function

Theorem 2. If X have an OGE-R distribution, then the moment generating function (mgf) of X, denoted as $M_x(t)$, is given by:

$$M_x(t) = \sum_{i=0}^{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{m=0}^{\infty}\binom{\alpha-1}{i}\binom{j}{k}(-1)^{i+j+k}(i+1)^j \frac{\alpha\lambda^{j+1}t^m\sigma^m 2^{\frac{m}{2}}}{j!m!(k-j-1)^{\frac{m+2}{2}}}\left(\frac{m}{2}\right)!$$

Proof. The mgf of OGE-R distribution is obtained using

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx \quad (20)$$

i.e, by replacing x^r with e^{tx} from (7) down to (16), we obtain the mgf of X as

$$M_x(t) = \sum_{i=0}^{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{\alpha-1}{i}\binom{j}{k}(-1)^{i+j+k}(i+1)^j \frac{\alpha\lambda^{j+1}}{j!\sigma^2}\int_0^\infty xe^{tx}e^{\frac{-(k-j-1)x^2}{2\sigma^2}}dx \quad (21)$$

But using series expansion

$$e^{tx} = \sum_{m=0}^{\infty}\frac{(tx)^m}{m!} \quad (22)$$

substituting equation (22) in (21), we get

$$M_x(t) = \sum_{i=0}^{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{m=0}^{\infty}\binom{\alpha-1}{i}\binom{j}{k}(-1)^{i+j+k}(i+1)^j \frac{\alpha\lambda^{j+1}t^m}{j!m!\sigma^2}\int_0^\infty xe^{\frac{-(k-j-1)x^2}{2\sigma^2}}dx \quad (23)$$

Now, we let

$$y = \left(\frac{(k-j+1)x^2}{2\sigma^2}\right)^{\frac{1}{2}} \Rightarrow x = \left(\frac{2y\sigma^2}{k-j-1}\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{(k-j-1)x}{\sigma^2} \Rightarrow dx = \frac{\sigma^2 dy}{(k-j-1)x}$$

Replacing $x$in the last expression and simplifying we have

$$dx = \frac{\sigma^2 dy}{(k-j-1)^{\frac{1}{2}}(2y\sigma^2)^{\frac{1}{2}}}$$

Substituting the above equations in (23), we get

$$M_x(t) = \sum_{i=0}^{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{m=0}^{\infty}\binom{\alpha-1}{i}\binom{j}{k}(-1)^{i+j+k}(i+1)^j \frac{\alpha\lambda^{j+1}t^m\sigma^m 2^{\frac{m}{2}}}{j!m!(k-j-1)^{\frac{m+2}{2}}}\int y^{\frac{m}{2}}e^{-y}dy \quad (24)$$

Using the gamma function and simplifying equation (24), we get

$$M_x(t) = \sum_{i=0}^{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{m=0}^{\infty}\binom{\alpha-1}{i}\binom{j}{k}(-1)^{i+j+k}(i+1)^j \frac{\alpha\lambda^{j+1}t^m\sigma^m 2^{\frac{m}{2}}}{j!m!(k-j-1)^{\frac{m+2}{2}}}\left(\frac{m}{2}\right)! \quad (25)$$

This completes the proof.

IV. RESULT

A. The Order Statistic

Suppose $X_1, X_2, \cdots, X_n$ is a random sample from a distribution with pdf $f(x)$and let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ denote the corresponding order statistic obtained from this sample. The pdf $f_{i:n}(x)$ of the i^{th} order statistic can be express as

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}f(x)F(x)^{i-1}[1-F(x)]^{n-i} \quad (26)$$

where $F(x)$ and $f(x)$ are the CDF and pdf of OGE-R distribution

Using the binomial expansion on $[1-F(x)]^{n-i}$ we get

$$[1-F(x)]^{n-i} = \sum_{r=0}^{n-i}\binom{n-i}{r}(-1)^r F(x)^r \quad (27)$$

Substituting equation (27) in (26) we get

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}f(x)\sum_{r=0}^{n-1}\binom{n-i}{r}(-1)^r F(x)^{r+i-1} \quad (28)$$

Substituting (5) and (6) in (28) we get

$$f_{i:n} = \sum_{r=0}^{n-i}\frac{n!(-1)^r}{(i-1)!(n-i-r)!r!}\frac{\alpha\lambda x}{\sigma^2}e^{\frac{x^2}{2\sigma^2}}e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)}(1-e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)})^{\alpha-1+\alpha(r+i-1)} \quad (29)$$

Simplifying equation (29) we get

$$f_{i:n}(x) = \sum_{r=0}^{n-i}\frac{n!(-1)^r}{(i-1)!(n-i-r)!r!}\frac{\alpha\lambda x}{\sigma^2}e^{\frac{x^2}{2\sigma^2}} e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)}(1-e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)})^{\alpha(i+r)-1} \quad (30)$$

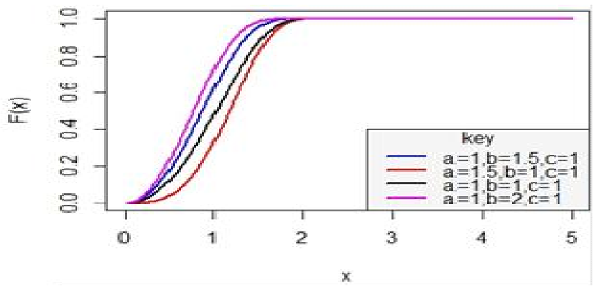

Figure 1: CDF of OGE-R distribution

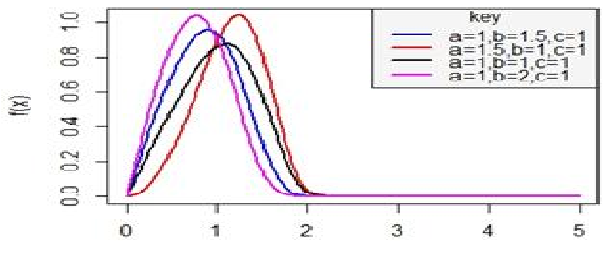

Figure 2: pdf of OGE-R distribution

B. Parameter Estimation

Let $\boldsymbol{X} = (X_1, X_2, \cdots, X_n)'$ be a sample of size n from the OGE-R distribution with parameters

where $\boldsymbol{\Theta} = (\alpha, \lambda, \sigma)$. Then the likelihood function L of this is given as;

$L = L(X_1, X_2, \cdots, X_n / \boldsymbol{\Theta})$

$$L = \prod_{i=1}^{n} \frac{\alpha\lambda x}{\sigma^2} e^{\frac{x^2}{2\sigma^2}} e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)} (1 - e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)})^{\alpha-1} \quad (31)$$

The log-likelihood function is

$$L = n\ln\alpha + n\ln\lambda - 2n\ln\sigma + \sum_{i=1}^{n} \ln x_i + \sum_{i=1}^{n} \frac{x_i}{2\sigma^2} + \lambda\sum_{i=1}^{n}(e^{\frac{x^2}{2\sigma^2}} - 1) + (\alpha - 1)\sum_{i=1}^{n} \ln(1 - e^{-\lambda(e^{\frac{x^2}{2\sigma^2}}-1)}) \quad (32)$$

The MLE's of α, λ and σ are the solutions of the following equations adopting numerical iterative method:

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\hat{\alpha}} + \sum \ln(1 - e^{-\lambda(e^{\frac{x_i^2}{2\sigma^2}}-1)}) = 0 \quad (33)$$

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\hat{\lambda}} + \sum_{i=1}^{n}(e^{\frac{x_i^2}{2\sigma^2}} - 1) + (\alpha - 1)\sum_{i=1}^{n}\{\frac{e^{-\hat{\lambda}(e^{\frac{x_i^2}{2\sigma^2}}-1)}(e^{\frac{x_i^2}{2\sigma^2}} - 1)}{(1 - e^{-\hat{\lambda}(e^{\frac{x_i^2}{2\sigma^2}}-1)})}\} = 0 \quad (34)$$

$$\frac{\partial L}{\partial \sigma} = \frac{-2n}{\hat{\sigma}} - \sum_{i=1}^{n}\frac{x_i^2}{\hat{\sigma}^3} + \frac{\lambda x_i^2}{\hat{\sigma}^3} e^{\frac{x_i^2}{2\hat{\sigma}^2}} + (\alpha - 1)\sum_{i=1}^{n}\{\frac{\frac{\lambda x_i^2}{\hat{\sigma}^3} e^{-\lambda(e^{\frac{x_i^2}{2\hat{\sigma}^2}}-1) + \frac{x_i^2}{2\hat{\sigma}^2}} (e^{\frac{x_i^2}{2\hat{\sigma}^2}} - 1)}{(1 - e^{-\lambda(e^{\frac{x_i^2}{2\hat{\sigma}^2}}-1)})}\} = 0 \quad (35)$$

These set of equations cannot be solved analytically and statistical software can be used to solve them numerically.

V. CONCLUSION

In this article, we have studied a new probability distribution called Odd Generalized Exponential-Rayleigh distribution, using the generator proposed by [7]. Some of its properties were obtained, the distribution of order statistics was derived. The parameters of the distribution were also obtained using the method of maximum likelihood.

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