A New Cubic Transmuted Exponential Distribution for Modelling Life-time data

A. A. Ogunde[1]; O. N. Eziche; B. Ajayi[3]; D. O. Omosigho

[1]Department of Statistics,
University of Ibadan, Nigeria.

[3]Department of Mathematics and Statistics,
Federal Polytechnic Ado-Ekiti, Nigeria.
E-mail: debiz95@yahoo.com [1]

***Abstract* — This article provides a new statistical distribution named Cubic Transmuted exponential distribution obtained using cubic transmutation map suggested by Aslam et al. (2018). Various statistical properties of this distribution were investigated which includes: Hazard function, moments, moment generating function, order statistics, Renyl entropy were obtained. The maximum likelihood estimator of the unknown parameters of the distribution was derived. Application to real data set shows it tractability over its sub-models in analyzing life data.**

Keywords - *Moment, Likelihood estimation, Moment generating function, Cubic Transmuted exponential distribution.*

I. Introduction

In recent time many attempts have been made to develop new generalised families of distribution. The common property of this distribution is that they are tractable, flexible and easily adaptable to various kinds of data. Such generalised distribution includes the work of Nofal, Z. M. et al. [8] developed the generalized transmuted-G family of distributions, Eugene et al. [6] developed family of beta-generated family of distribution. Cordeiro and Castro [5] proposed the Kumaraswamy-G family of distribution.

The Weibull-G family of probability distribution was developed by Bourguignon et al. [3]. The Topp-Leone-G family of distribution was introduced by Youssuf H. M et al [12]. Granzotto et al. [6] Aslam Mohammad et al. [7], Rahman et al. [14] studied the Generalized Cubic Transmuted G-distribution using different approaches that found it root from The Generalised transmuted family of distribution by Shaw et al. [9]. This work generalizes the exponential distribution using the generalized Cubic Transmuted-G distribution to impute flexibility and tractability into the exponential distribution.

1.1 Aim and objectives of the research

The purpose of this work is to generalize the exponential distribution to obtain a new distribution named Cubic Transmuted Exponential (CTE) distribution which can be used to model bi-modal data with the aim of achieving the following objectives:

- To derive the probability density function and the cumulative density function of the new distribution.
- To determine some structural properties of the proposed distribution this includes: Moment, moment generating function, Renyl entropy and order statistics.
- To determine the flexibility and tractability of the new distribution with application to a real data set.

1.2 Exponential distribution

The exponential (E) distribution plays an important role in lifetime testing. In probability theory and statistics, the exponential distribution can be said to be probability distribution of the time between events govern by poisson point process. That is, it can be described as a process in which events occur continuously and independently at a constant average rate. The exponential distribution is measures the amount of time until some specific event occurs.

A random variable X is said to follow an exponential distribution if its cumulative distribution function is given by

$$G(x,\gamma) = 1 - e^{-\gamma x} \tag{1}$$

where the variable x and the parameter γ is positive real quantities. The exponential distribution occur in many different connections such as the radioactive or particle decays or the time between events in a Poisson process where events happen at a constant rate.

The graph of the hazard function is drawn below in figure 1 for various values of the parameter.

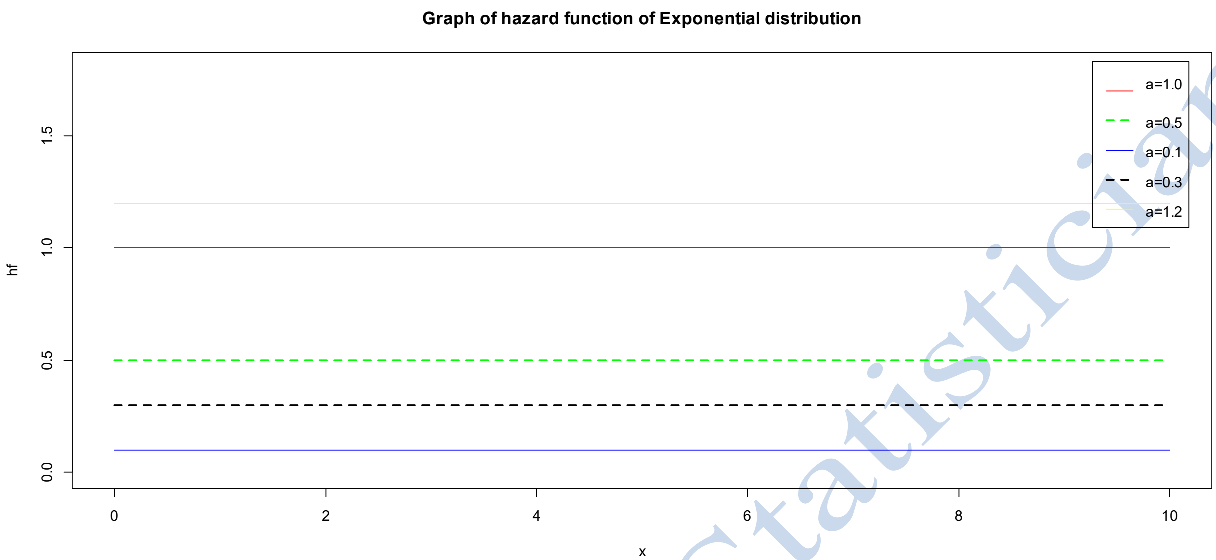


Figure 1: The Graph of the hazard function of exponential distribution

Figure 1 shows that the exponential distribution exhibits a constant rate failure which is not realist in real life phenomenon, therefore is need for re-modification.

II. METHODOLOGY

2.0 Generalised Transmuted Family of Distributions

A general transmuted family of distributions that can be used to generate new families. Let X be a random variable with cdf $\bar{H}(x)$, then a general transmuted family; called k-transmuted family; is defined by

$$\bar{H}(x) = G(x) + [1 - G(x)] \sum_{i=1}^{k} \phi_i [G(x)]^i \qquad (2)$$

with $\phi_i \in [-1,1]$ for $i = 1,2,. . .,p$ and $-k \leq \sum_{i=0}^{p} \lambda_i \leq 1$. The general transmuted family reduces to the base distribution for $\phi_i = 0$, for $i = 1,2,3,. . .,p$. The density function corresponding to (2) is

$$\bar{h}(x) = g(x)\left\{1 - \sum_{i=1}^{k} \phi_i [G(x)]^i + [1 - G(x)] \sum_{i=1}^{k} \phi_i G^i(x)\right\} \qquad (3)$$

The density function of the quadratic transmuted quadratic family of distributions which was obtained by letting $p = 1$ as defined by Shaw et al. [11], is given as

$$\bar{H}(x) = (1 + \phi)G(x) - \lambda G^2(x) \qquad (4)$$

Where $\phi \in (1, -1)$ is the transmutation parameter. The quadratic transmuted family of distribution given in (4) has a wider area of applications to any baseline $G(x)$. The quadratic transmuted distribution does not provide a reasonable fit if the data is bi-modal in nature.

To address the problem of bi-modality encountered in real data, the cubic transmuted family of distributions is obtained by setting $k = 2$ in (2), which was demonstrated in the work of Aslam Mohammad et al.[1], and is given as

$$\bar{H}(x) = (1 + \phi_1)G(x) + (\phi_2 - \phi_1)G^2(x) + (1 - \phi_2)G^3(x) \qquad (5)$$

Where $\phi_1 \in [-1,1]$, $\phi_2 \in [-1,1]$ are the transmutation parameters and obey the condition
$-2 \leq \phi_1 + \phi_2 \leq 1$.

The pdf corresponding to the equation (5) is defined as

$$\bar{h}(x) = g(x)[\phi_1 + 2(\phi_2 - \phi_1)G(x) + 3(1 - \phi_2)G^2(x)] \qquad (6)$$

This method has been used by several authors to generalize a distribution for better adaptability to data. For examples Ogunde et al. [9], Rahman et al. [14], Hussein Eledum [7].Bugra Sracoglu and Caner Tams [3]

Cubic Transmuted Exponential distribution

A random variable X is said to follow the CTE distribution if its cdf is given by

$$\bar{H}(x;\omega) = [\phi_1(1-e^{-\gamma x}) + (\phi_2-\phi_1)(1-e^{-\gamma x})^2 + (1-\phi_2)(1-e^{-\gamma x})^3] \quad (7)$$

and the associated pdf is given by

$$\bar{h}(x;\omega) = \gamma e^{-\gamma x}[\phi_1 + (\phi_2-\phi_1)(1-e^{-\gamma x}) + (1-\phi_2)(1-e^{-\gamma x})^2] \quad (8)$$

Figure 2.0 shows the shape of the density function of the CTE distribution from arbitrary values of the parameters. The graph shows that the pdf of CTE is unimodal and right skewed with different degrees of kurtosis.

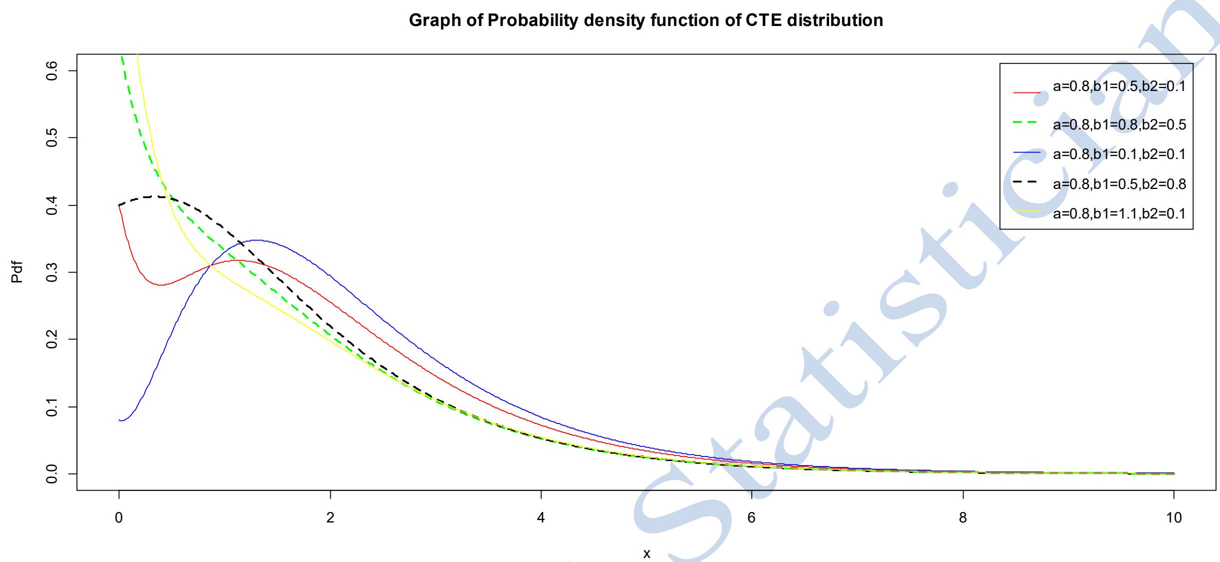


Figure 2: The Graph of the density function of CTE distribution

Figure 2.0 drawn above illustrates the properties of CTE distribution being capable in handling both the unimodal and bimodal data.

The survival and the hazard rate function of the ECTE distribution are respectively given by

$$H(x;\omega) = 1 - [\phi_1(1-e^{-\gamma x}) + (\phi_2-\phi_1)(1-e^{-\gamma x})^2 + (1-\phi_2)(1-e^{-\gamma x})^3] \quad (9)$$

and

$$h(x;\omega) = \frac{\gamma e^{-\gamma x}[\phi_1 + (\phi_2-\phi_1)(1-e^{-\gamma x}) + 3(1-\phi_2)(1-e^{-\gamma x})^2]}{1 - [\phi_1(1-e^{-\gamma x}) + (\phi_2-\phi_1)(1-e^{-\gamma x})^2 + (1-\phi_2)(1-e^{-\gamma x})^3]} \quad (10)$$

Figure 3 and figure 4 respectively are the graph of the survival function and hazard function of CTE distribution for various values of the parameters. Figure 4 indicates that the hazard function of the CTE distribution exhibits the upside down bathtub failure rate for the values of the parameters considered.

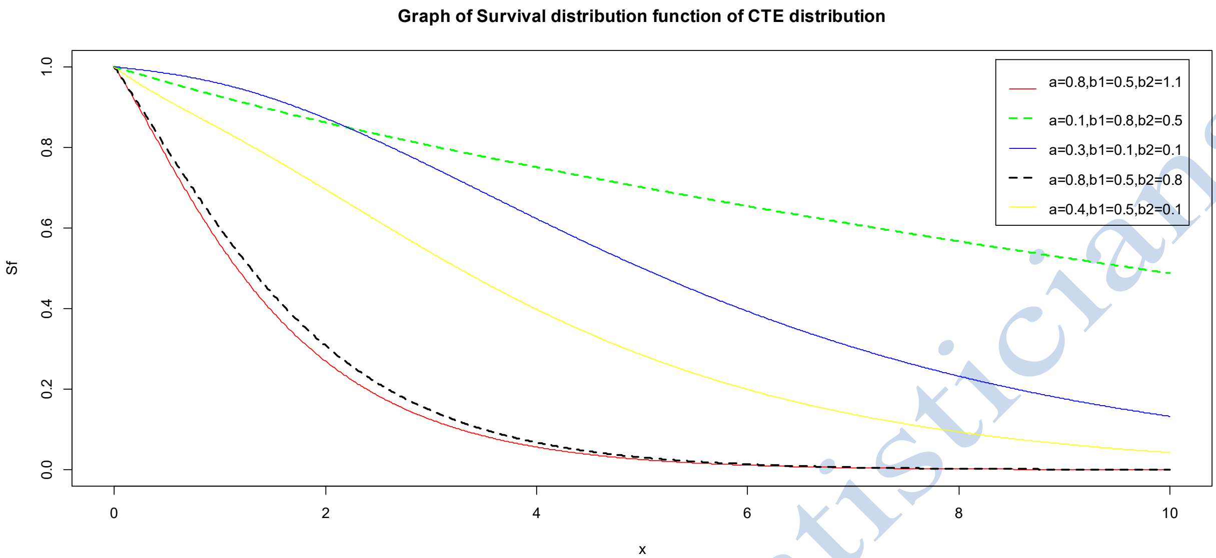


Figure 3: The Graph of the survival function of CTE distribution

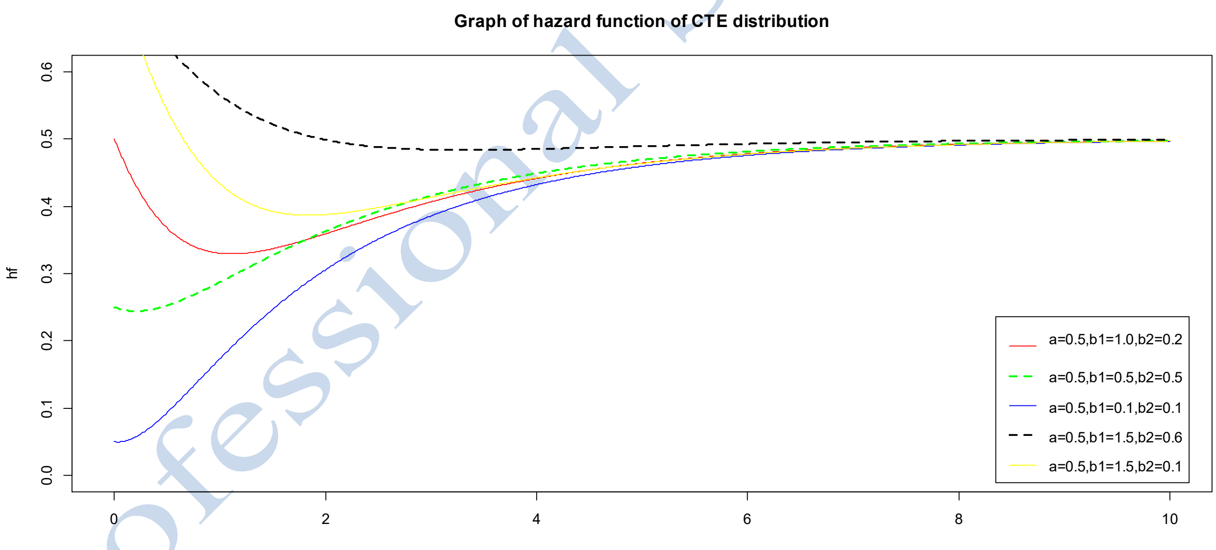


Figure 4: The Graph of the hazard function of CTE distribution

2.1 Moments

Moment of a distribution plays a very important role in statistical analysis. They are used for estimating features and characteristics of a distribution such as skewness, kurtosis, measures of central tendency and measures of dispersion.

LEMMA 3.1. $If\ X \sim CTE(\omega)$, where $\omega = \{\gamma, \phi_1, \phi_2\}$, then the r^{th} non-central moment of X is given by

$$E(X^r) = \mu_r' = \gamma^{-r}\Gamma(r+1)[1-(\phi_2-\phi_1)(1-2^{-r-1}) + (1-\phi_2)(1-2^{-r-1}+3^{-r-1}]$$

Proof.: By definition, the $\boldsymbol{r^{th}}$ non-central moment is given by

$$\mu_r' = \int_0^\infty x^r f(x;\omega)dx \qquad (11)$$

Putting (8) in (11), we have

$$\mu_r' = \gamma \int_{-\infty}^{\infty} x^r\, e^{-\gamma x}\, [\lambda_1 + (\lambda_2 - \phi_1)(1-e^{-\gamma x}) + (1-\phi_2)(1-e^{-\gamma x})^2]\, dx \qquad (12)$$

From equation (12), taking

$$W_1 = \phi_1 \gamma \int_0^\infty x^r\, e^{-\gamma x}\, dx$$
$$= \gamma^{-r}\Gamma(r+1) \qquad (13)$$

$$W_2 = (\phi_2 - \phi_1) \int_{-\infty}^{\infty} x^r\, e^{-\gamma x}\, (1-e^{-\gamma x})dx$$
$$= (\phi_2 - \phi_1)\gamma^{-r}(1-2^{-r-1})\Gamma(r+1) \qquad (14)$$

$$W_3 = (1-\phi_2) \int_{-\infty}^{\infty} x^r\, e^{-\gamma x}\, (1-e^{-\gamma x})^2$$
$$= (1-\phi_2)\gamma^{-r}(1-2^{-r}+3^{-r-1})\Gamma(r+1) \quad (15)$$

Combining equation (13), (14) and (15), we obtain the r^{th} moment of CTE distribution given as:

$$\mu_r' = \gamma^{-r}\Gamma(r+1)[1-(\phi_2-\phi_1)(1-2^{-r-1}) + (1-\phi_2)(1-2^{-r-1}+3^{-r-1}] \qquad (16)$$

For $r = 1, 2, \ldots$, $\Gamma(.)$ is the gamma function. Table 1 and 2 drawn below gives the first four moments, variance($\boldsymbol{\delta^2}$), Coefficient of Variation(CV), Skewness (S) and kurtosis (K) The values for $\boldsymbol{\delta}^2$ and CV, are respectively given by

Table 1: first four moments, $\boldsymbol{\delta^2}$, $\boldsymbol{CV}$, $\boldsymbol{S}$ and $\boldsymbol{K}$ ($\boldsymbol{\phi_2 = 0.5, \phi_1 = 0.5}$)

γ	μ_1'	μ_2'	μ_3'	μ_4'	δ^2	CV	S	K
0.5	2.8333	13.4444	87.8889	734.3704	5.4168	0.8215	1.5152	6.5619
1.0	1.4167	3.3611	10.9861	45.8982	1.3541	0.8214	1.5154	6.5624
1.5	0.9444	1.4938	3.2661	9.0663	0.6019	0.8215	1.5386	6.4472
2.0	0.7083	0.8403	1.3733	2.8686	0.3386	0.8215	1.5147	6.5600
2.5	0.5667	0.5378	0.7031	1.1750	0.2165	0.8210	1.5166	6.5728
10.5	0.1349	0.0305	0.0095	0.0038	0.0123	0.8221	1.5148	6.6794
100.5	0.0141	0.0003	1.0816e-5	4.5075e-7	0.0001	0.7134	3.7327	8.0009

From table 1.0 drawn above we can conclude that the CTE distribution is positively skewed and kleptokurtic. The variance decreases tending to zero as the value of γ increases and the CV is relatively constant except for a large value of γ.

Table 2: first four moments, $\boldsymbol{\delta^2}$, CV $\boldsymbol{\delta^2}$, $\boldsymbol{CV}$, $\boldsymbol{S}$ and $\boldsymbol{K}$. ($\boldsymbol{\phi_2 = -0.5, \phi_1 = -0.5}$)

γ	μ_1'	μ_2'	μ_3'	μ_4'	δ^2	CV	S	K
0.5	4.50	24.3333	167.6667	1435.1111	4.0833	0.4491	2.5956	8.6017
1.0	2.25	6.0833	20.9583	89.6944	1.0208	0.4490	2.5959	8.6017
1.5	1.50	2.7037	6.2099	17.7174	0.4537	0.4491	2.5957	8.6009
2.0	1.125	1.5208	2.6198	5.6059	0.2552	0.4490	2.5966	8.5979
2.5	0.90	0.9733	1.3413	2.2962	0.1633	0.4490	2.5972	8.6046
10.5	0.2143	0.0552	0.0181	0.0074	0.0093	0.4500	2.5591	8.8761
100.5	0.0224	0.0006	2.0629e-5	8.8146e-7	0.0001	0.4464	2.7878	8.4148

From table 2.0 drawn above we can conclude that the CTE distribution is positively skewed and kleptokurtic. The variance decreases tending to zero as the value of γ increases and the CV is relatively constant except for a large value of γ.

2.2 Moment generating function

Moment Generating Functions (MGF): These are special functions that are used to obtain the moments and its functions such as: mean and variance of a random variable in a simpler way. LEMMA 3.2. If $X \sim ECTE\ (\omega)$, where $\omega = \{\gamma, \phi_2, \phi_1\}$, then the MGF of X is given by

Proof: By definition, the MGF is given by

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} f(x;\omega)dx \qquad (17)$$

Using the series expansion e^{tx} gives

$$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_0^\infty x^k f(x)dx = \sum_{k=0}^{\infty} \frac{t^k \mu_k'}{k!} \qquad (18)$$

Putting equation (16) in (18), we have

$$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \gamma^{-r} \Gamma(r+1)[1 - (\phi_2 - \phi_1)(1 - 2^{-r-1}) + (1 - \phi_2)(1 - 2^{-r-1} + 3^{-r-1}]$$

2.3 Entropy

Entropies is a measure of randomness of a system and has been used extensively in information theory. Two popular entropy measures are Renyi entropy [10] and Shannon entropy [13]. A large value of the entropy indicates a greater uncertainty in the data. The Shannon entropy is a special case of the Renyi entropy when $z \to 1$ and is given by

$E[-\log(f(x;\omega))]$.

LEMMA 3.3: If the random variable X has a CTE distribution, then the Renyi entropy of X is given by

$$E_R(z) = \frac{1}{1-z} \log\left[z\lambda^{z-1} \int_0^\infty \{\phi_1 + (\phi_2 - \phi_1)(1 - e^{-\gamma x}) + (1 - \phi_2)(1 - e^{-\gamma x})^2\}^z dx\right] \qquad (19)$$

Proof: By definition

$$E_R(z) = \frac{1}{1-z} \log\left\{\int_0^\infty f^z(x)dx\right\}, z > 0, z \neq 0. \qquad (20)$$

Putting equation (8) in (20), we have

$$E_R(z) = \frac{1}{1-z} \log\left\{\int_0^\infty \{\gamma e^{-\gamma x} [\phi_1 + (\phi_2 - \phi_1)(1 - e^{-\gamma x}) + (1 - \phi_2)(1 - e^{-\gamma x})^2]\}^z dx\right\}$$

The equation above can be written as

$$E_R(z) = \frac{1}{1-z} \log\left\{\int_0^\infty I_1{}^z dx * \int_0^\infty I_2{}^z dx\right\} \qquad (21)$$

where

$$\int_0^\infty I_1{}^z dx = \int_0^\infty (\gamma e^{-\gamma x})^z dx = z\lambda^{z-1}$$

Finally we have,

$$E_R(z) = \frac{1}{1-z} \log\left[z\lambda^{z-1} \int_0^\infty \{\phi_1 + (\phi_2 - \phi_1)(1 - e^{-\gamma x}) + (1 - \phi_2)(1 - e^{-\gamma x})^2\}^z dx\right]$$

2.4 Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from CTE distribution and let $Z_{j:n}$ denote the j^{th} order statistics, then the pdf of $x_{j:n}$ is given by

$$f_{j:n}(x) = \frac{n!}{(i-1)!\,(n-i)!} \bar{h}_x(z)[\bar{H}_x(z)]^{j-1}[1 - \bar{H}_x(z)]^{n-j} \qquad (22)$$

Applying binomial expansion in (22), we have

$$\bar{h}_{j:n}(x) = \frac{n!}{(i-1)!\,(n-i)!} \bar{h}_x(z) \sum_{k=0}^{\infty} \binom{n-j}{k} [\bar{H}_x(z)]^{j+k-1} \qquad (23)$$

Using equation (7) and (8) in (23), we obtain the j^{th} order statistics of CTE distribution given by

$$\bar{h}_{j:n}(x) = \frac{n!\,\gamma W_x}{(i-1)!\,(n-i)!} e^{-\gamma x} [\phi_1 + (\phi_2 - \phi_1)(1 - e^{-\gamma x}) + (1 - \phi_2)(1 - e^{-\gamma x})^2] \qquad (24)$$

where

$$W_x = \sum_{k=0}^{\infty} \binom{n-j}{k} \left[[\phi_1(1 - e^{-\gamma x}) + (\phi_2 - \phi_1)(1 - e^{-\gamma x})^2 + (1 - \phi_2)(1 - e^{-\gamma x})^3]\right]^{j+k-1}$$

We can obtain the first orders statistics by taking $j = 1$, also we can obtain the n^{th} order by taking $j = n$ in equation (24)

3.0 Maximum likelihood estimation for parameters of CTE distribution

Let $X_1, X_2, \ldots, X_n$ be a random sample taken from CTE with parameters $\omega(\gamma, \lambda_1, \lambda_2)$. The likelihood function is given by

$$L(Z\,|\,\underline{x}) = \prod_{i=1}^{n} \gamma e^{-\gamma x}\{\phi_1 + 2(\phi_2 - \phi_1)(1 - e^{-\gamma x}) + 3(1 - \phi_2)(1 - e^{-\gamma x})^2\} \qquad (25)$$

And the log-likelihood function is given by

$$l(Z\,|\,\underline{x}) = n\log(\gamma) - \gamma \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \log\{\phi_1 + 2(\phi_2 - \phi_1)\kappa_i + 3(1-\phi_2){\kappa_i}^2\} \qquad (26)$$

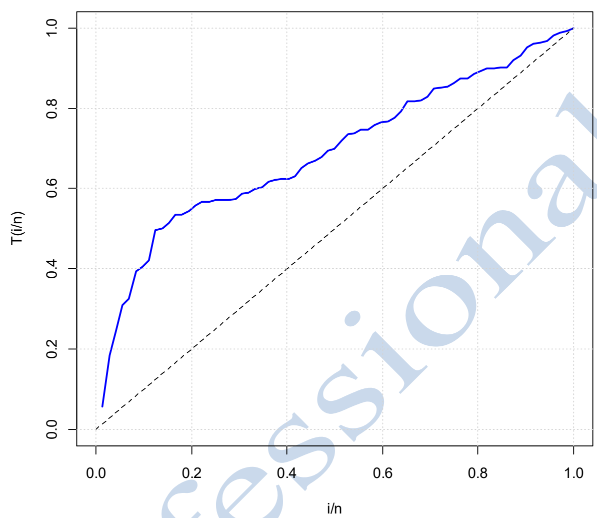
where, $\kappa_i = 1 - e^{-\gamma x}$
To obtain a numerical solution for the values of the estimates of CTE distribution we may employ software such as R, Maple, OX Program etc.

III. Application

In this section, we present one example to demonstrate the flexibility and the applicability of the CTE distribution in modelling real world data. We fit the density functions of the CTE distribution and its sub-models to two real life data. The data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal et al. [2]. Table 3 gives the exploratory data analysis of the data and Table 4.0 gives the MLEs of the distributions and the values for the measure of fit. Fig.5 gives the graph of the Total Time on Test plot and the graph of the kernel density of the pig data.

Table 3. Exploratory Analysis of the data

Min.	Q_1	Med.	Mean	Q_3	Max.	Var.	Skew.	Kurt.
0.100	1.080	1.495	1.768	2.240	5.550	1.070	1.371	2.225



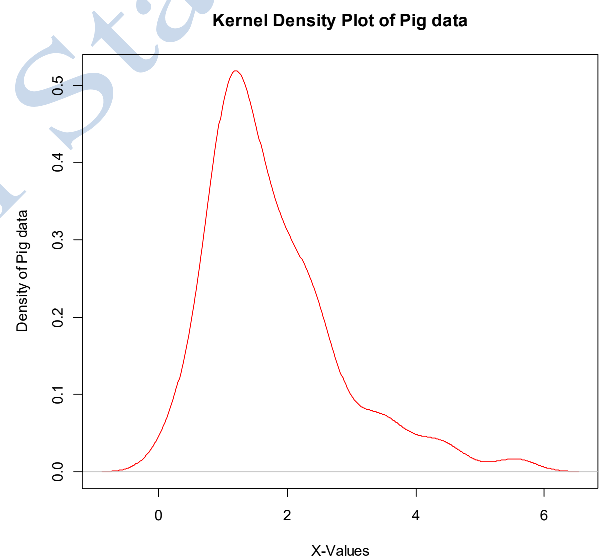


Figure 5: TTT plot and Kernel density plot for the pig data

Table 4. MLEs and the measures of goodness of fit

Model	Parameters	−l	AIC	BIC	CAIC	HQIC
CTE	$\lambda_1 = 1.1506(0.0945)$ $\lambda_2 = 0.2671(0.1420)$ $\gamma = -0.9734(0.3583)$	92.904	191.808	198.639	192.162	194.528
E	$\gamma = 0.5655(0.0666)$	113.037	228.074	230.351	228.131	228.980

The ML estimates along with their standard error (SE) of the model parameters are provided in In the same tables, the analytical measures including; minus log-likelihood(-log L), Akaike information Criterion (AIC), Consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC). The fit of the proposed CTE distribution is compared its sub-model.

IV. Conclusion

The cubic transmuted exponential distribution shows a reasonable fit with the life data than exponential distribution in terms of the values of AIC, BIC, CAIC, and HQIC. The the distribution is positively skewed and kleptokurtic and the coefficient of variation is relatively constant.

References

[1] Aslam Muhammad et al. (2017). Cubic Transmuted-G Family of distributions and its Properties. Stochastic and quality Control, De Gruyter, Vol. 33(2), pages 103-112.

[2] Bjerkedal, T. et al. (1960). Acquisition of resistance in guinea pigs infected with differentdoses of virulent tubercle bacilli. American Journal of Hygiene, 72(1):130-148.

[3] Bourguignon, M., Silva, R. B. and Cordeiro, G. M. (2014). The Weibull-G Family of Probability distributions. Journal of Data Science, 12: 53-68.

[4] Bugra Sracoglu and Caner Tams (2018). A new statistical distribution: cubic rank transmuted kumaraswamy distrihbution and its properties. J.Natn. Sci.Foundation Sri Lanka,. 46(4): 505-518.

[5] Cordeiro, G. and Castro, M. (2010). A new family of generalized distributions. *Journal of Statistical Computation and Simulation*, 81:883-898.

[6] Eugene, N., Lee, C., and F., F. (2002). Beta-normal distribution and its applications. Communications in Statistics-Theory and Methods, 31:497-512.

[7] Hussein Eledum. (2020). Some cubic exponential pareto-I distribution. Journal of mathematics and statistics.

[8] Nofal, Z. M., Afify, A. Z., Yousof, H. M., and Cordeiro, G. M. (2017). The generalized transmuted-G family of distributions. Communications in Statistics-Theory and Methods, 46:4119{4136, doi:10.1080/03610926.2015.1078478.

[9] Ogunde A. A. et al. (2020). Cubic transmuted Gompertz distribution: Properties and Applications. Journal of Advances in Mathematics and Computer Science 35(1):105-116, 2020.

[10] Renyil, A. L. (1961). On Measure on Entropy and Information. In fourth Berkeley symposium on mathematical statistics and probability. (Vol. 1, pp. 547-561).

[11] Shaw, W. T. and Buckley, I. R. C. (2007). The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. UCL discovery repository, http://discovery.ucl.ac.uk/id/eprint/643923

[12] Yousof, H.M et al.20017.The transmuted Topp-leone G-family of distribution: theory, characterization and applications. *Journal of Data Science* **15**: 723-740.

[13] Shannon, C. E. (1948). A Mathematical theory of comminication. The Bell System Technica;l Journal, 27: 379-423.

[14] Rahman MM, Al-Zahrani B, Shahbaz MQ. A general transmuted family of distributions. Pakistan Journal of Statistical Operation Research. 2018;14:451–469.