Specification of GARCH Model in The Presence of Asymmetric Innovations

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***Abstract*—An empirical analysis of the mean return and conditional variance of Nigeria Stock Exchange (NSE) index is performed using various error innovations in GARCH models. Conventional GARCH model which assumed normal error term failed to capture volatility clustering, leptokurtosis and leverage effect as a result of zero skewness and kurtosis respectively. We re-modify error distributions of GARCH (p,q) model inference using some thick-tailed distributions. Method of Quasi – Maximum Likelihood Estimation (MLE) was used in parameter estimation. The robust model that explained the NSE index is determined by loglikelihood and model selection Criteria. Our result shows that GARCH model with fat-tailed densities improves overall estimation for measuring conditional variance. The GARCH model using Beta-Skewed-t distribution is the most successful for forecasting NSE index.**

Keywords: *GARCH , Nigeria Stock Index , Maximum Lilkelihood Estimation (MLE) and Beta Skewed -t distributions.*

I. Introduction

Volatility clustering and leptokurtosis are commonly observed in financial time series [1]. Another phenomenon often encountered is the so called leverage effect [9], which occurs when stocks prices change are negatively correlated with changes in volatility. Observation of this type in financial time series have led to the use of a wide range of varying variance models to estimate and predict volatility. In his seminar paper, Engle [4] proposed to model time-varying conditional variance with Autoregressive Conditional Heteroskedasticity (ARCH) processes using lagged disturbances; Empirical evidence based on his work showed that a high ARCH order is needed to capture the dynamic behaviour of conditional variance. The Generalized ARCH (GARCH) model of Bollerslev [1] fulfills this requirement as it is based on an infinite ARCH specification which reduces the number of estimated parameters from infinity to two.

Both the ARCH and GARCH models capture volatility clustering and Leptokurtosis, but as their distribution is symmetric.Another problem encountered when using GARCH models is that they do not always fully embrace the thick tails property of high frequency financial times series. To overcome this drawback Bollerslev [1], Baille and Bollerslev [12] and Beine et al [13] have used the Student's t- distribution. Similarly to capture skewness. Liu and Brorsen [14] used an asymmetric stable density. To model both skewness and kurtosis Fernandez and Steel [15] used the skewed Student's t-distribution.Pagan and Schwert [16] and Lundo et al [17] carried out a comprehensive studied on forecasting conditional variance with asymmetric GARCH models. A comparison of normal density with non-normal ones was made by Baillie and Bollerslev[12], Lambert and Laurent [18], Jun Yu[19] and Siourounis [20].

The disadvantage of the normal GARCH (1,1) model is that the conditional excess kurtosis is zero, and both unconditional and conditional skewness are zero, thus, volatility clustering , leverage effect and leptokurtosis cannot be capture adequately. This work intends to re-modify error distributions of GARCH (p, q) model inference under violation of normality in favour of some thick-tailed distributions. A comparison between symmetric and asymmetric distributions was carried out using four different density functions. We investigate the forecasting performance of GARCH model together with different density functions: normal distribution, Student's -t distribution, Generalized Error distribution and Generalized Beta Skewed-t distribution. We forecast Nigeria Stock Exchange (NSE) index [5], we use several standard performance measurements and our results suggest that one can improve overall estimation by using asymmetric GARCH model with fat-tailed densities for measuring conditional variance.

This paper is structure as follows. Section II presents the data. In Section 3.0 to 3.3, we present Methodology of the GARCH models and estimation procedures used in the article. In Section 4.0, we present the results, tables and forecasting results.

II. Materials and Methods

The data consist of 180 monthly observations of the NSE Stock Index [5] from period January 2000 to December 2015 which was obtained from statistical Central Bank of Nigeria Bulletins 2016. To estimate and forecast this index, we use GARCHFIT in R package. Initially the assets prices are transformed into log return series, R_t given by

$$R_t = \log Y_t - \log Y_{t-1} = log\frac{Y_t}{Y_{t-1}} = log\left(1 + \frac{Y_t - Y_{t-1}}{Y_{t-1}}\right) \quad (1)$$

where Y_t is All Share Index (ASI) for day t.In this section we review the different ARCH models used in the paper, let$R_t$be a time series of assest returns whose mean equation is

$$R_t = a_0 + \sum_{i-1}^{s} a_i R_{t-i} + \varepsilon_t \quad (2)$$

where R_{t-i} *is* the information available at t-i and ε_t are the random innovations which follows Normal , Student-t , GED and newly proposed Generalized Beta Skew –t Distributions.

To explain the conditional Heteroskedasticity dynamic, Engle [4] proposed the Autoregressive Conditional Heteroskedasticity (ARCH) model that estimate the variance of returns as a simple quadratic functions of the lagged values of the innovations:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-1}^2 \quad (3)$$

where $\epsilon_t = \sigma_t z_t$ and z_t is an iid variable with mean zero and variance one.

One weakness of ARCH model is that it often requires many parameters and a high order of q to capture the volatility process. Bollerslev [1] propose a Generalized ARCH model based on an infinite ARCH Specification that enable us to reduce the number of estimated parameters by imposing restrictions. The standard GARCH (p,q) model express the variance at time t, σ_t^2as

$$\sigma_t^2 = \alpha_0 + \sum_{t=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{t=1}^{p} \beta_i \sigma_{t-i}^2 \quad (4)$$

where $\alpha_0 > 0$, $\alpha_i \geq 0$ (for i = 1,– – –, q), $\beta_i \geq 0$(for j=1, …, p) ,is sufficient for conditional variance to be positive. Where

$\alpha_i(L) = \sum_{t=1}^{q} \alpha_i L^i$and $\beta_j(L) = \sum_{t=1}^{p} \beta_i L^j$

If all the roots of the polynomial equal to zero lies outside the unit circle, we have

$$\sigma_t^2 = \alpha_0[1 - \beta(L)]^{-1} + \alpha(L)[1 - \beta(L)]^{-1}\varepsilon_t^2 \quad (5)$$

We also consider GARCH (1,1) model as the variant model ,which is

$$\sigma_t^2 = \alpha_0 + \alpha_1\varepsilon_{t-1}^2 + \beta_1\sigma_{t-1}^2 \quad (6)$$

The generalized beta distribution of the first kind was introduced by McDonald [6], with link function

$$g(y) = \frac{c}{\beta(a.b)}[F(y)^{ac-1}(1 - (F(y)^c)^{b-1}]f(y) \quad (7)$$

where $F(x) = I_{x(a,b,c)}$ is an incomplete beta function and f(y) is the probability function of student-t. A random variable y is said to be a Generalized Beta Skewed –t distribution iff$f(y_t: a, b, c) =$

$$\frac{c}{\beta(a.b)} I^{ac-1}(1 - I^c)^{b-1} \frac{\gamma\left(\frac{v+1}{2}\right)}{\gamma\left(\frac{v}{2}\right)\sqrt{\pi(v-2)\sigma_t^2}}\left[1 + \frac{\varepsilon_t^2}{\sigma_t^2}\frac{1}{v-2}\right]^{-\frac{v+1}{2}} \quad (8)$$

If we assume that$\varepsilon_t \sim GBt\ (v, \mu, \sigma, a, b, c)$, we have

$$f(y_t: a, b, c) = \frac{c}{\beta(a.b)} I^{ac-1}(1 - I^c)^{b-1} \frac{\gamma\left(\frac{v+1}{2}\right)}{\gamma\left(\frac{v}{2}\right)\sqrt{\pi(v-2)\sigma_t^2}}\left[1 + \frac{\varepsilon_t^2}{\sigma_t^2}\frac{1}{v-2}\right]^{-\frac{v+1}{2}} \quad (9)$$

Considering mean equation as AR (1) model and variance equation as GARCH (1,1) model then the log likelihood when the error term follows generalized beta Skew t distribution;

$$l = logc - n[log\gamma(a) + log\gamma(b) - log\gamma(a+b)] + nlog\gamma\left(\frac{v+1}{2}\right) - nlog\gamma\left(\frac{v}{2}\right) - \frac{n}{2}\left(log\pi + \log(v-2) + \sum_{t=2}^{n} log\,\alpha_0 + \alpha_1\varepsilon_{t-1}^2\right) + (ac-1)\sum_{t=2}^{n} logI + \sum_{t=2}^{n}\log(1 - I^c) - \left(\frac{v+1}{2}\right)\sum_{t=2}^{n} log\left[1 + \frac{(Y_t - \emptyset_1 Y_{t-1})^2}{\alpha_0 + \alpha_1\varepsilon_{t-1}^2(v-2)}\right] \quad (10)$$

Differentiate$l(\theta)$and equate to zero with rest to $\theta^\tau = (a, b, c, \alpha_0, \alpha_1, \emptyset_1)$, let i= prime in equation 11 and 12,

we have

$$\frac{\partial l}{\partial a} = -n\frac{n\gamma(a)^i}{\gamma(a)} + n\frac{\gamma(a+b)^i}{\gamma(a+b)} + c\sum_{t=2}^{n} logI \quad (11)$$

$$\frac{\partial l}{\partial b} = -n\frac{n\gamma(b)^i}{\gamma(b)} + n\frac{\gamma(a+b)^i}{\gamma(a+b)} + c\ \sum_{t=2}^{n} \log(1-I) \qquad (12)$$

$$\frac{\partial l}{\partial c} = \frac{n}{c} + a\ \sum_{t=2}^{n} logI + (b-1)\sum_{t=2}^{n} \frac{\frac{\partial}{\partial c}I^c}{1-I^c} \qquad (13)$$

$$\frac{\partial l}{\partial \alpha_0} = \sum_{t=2}^{n} \frac{1}{\alpha_0+\alpha_1(|\varepsilon_{t-1}|-\gamma_1\varepsilon_{t-1})^2} \qquad (14)$$

$$\frac{\partial l}{\partial \alpha_1} = \sum_{t=2}^{n} \frac{(|\varepsilon_{t-1}|-\gamma_1\varepsilon_{t-1})^2}{\alpha_0+\alpha_1(|\varepsilon_{t-1}|-\gamma_1\varepsilon_{t-1})^2} \qquad (15)$$

$$\frac{\partial l}{\partial \gamma_1} = -2\alpha_1 \sum_{t=2}^{n} \frac{\varepsilon_{t-1}}{\alpha_0+\alpha_1(|\varepsilon_{t-1}|-\gamma_1\varepsilon_{t-1})^2} \qquad (16)$$

$$\frac{\partial l}{\partial \emptyset_1} = (v+1)\sum_{t=2}^{n} \frac{Y_{t-1}(Y_t-\emptyset_1 Y_{t-1})}{\left[1+\frac{(Y_t-\emptyset_1 Y_{t-1})^2}{\alpha_0+\alpha_1(|\varepsilon_{t-1}|-\gamma_1\varepsilon_{t-1})^2(v-2)}\right]} (17)$$

Other Existing functional distributions considered in the literature are Normal, Student-t and GED Innovation [3] [10] . The log- likelihood from normal distribution is

$$l_t = -\frac{1}{2}\left[nlog(2\pi) + \sum_{i=1}^{n}\frac{\varepsilon_t{}_t^2}{\sigma_t^2} + \sum_{i=1}^{n} log\sigma_t^2\right](18)$$

$z_t = \frac{\varepsilon_t}{\sigma_t}$ where $\varepsilon_t = z_t$ is the GARCH time series innovations and n is the sample size of the time series. For Student t-distribution, we have

$$l_t = -\frac{1}{2}\left[nlog\left[\frac{\pi(v-2)\gamma\left(\frac{v}{2}\right)^2}{\gamma(\pi\frac{(v+1)}{2})^2}\right]\right] + \sum_{i=1}^{n} log\sigma_t^2 + (v+1)\sum_{i=1}^{n} log\left(1+\frac{\varepsilon_t{}_t^2}{\sigma_t^2(v-1)}\right)$$
(19)

Where v is the degree of freedom to be estimated and Γis the gamma function. For GED, it is

$$l_t = -\frac{1}{2}\left[nlog\left[\frac{\gamma(v-1)^3}{\gamma(3v^{-1})\gamma\left(\frac{v}{2}\right)}\right]\right] + \sum_{i=1}^{n} log\sigma_t^2 + (v+1)\sum_{i=1}^{n} log\left(\frac{\gamma(3v^{-1})\varepsilon_t{}_t^2}{\sigma_t^2\gamma(v^{-1})}\right)(20)$$

where v is the tail thickness parameter. The log-likelihood functions in the 10, 18, 19 and 20 are simplified using R code.

I. ANALYSIS

To obtain a stationary series, we use the returns $R_t = 100(\log(Y_t) - \log(Y_{t-1}))$ where Y_t is the closing value of index at month t. The sample statistics for the returns R_t are exhibited in table 1. For NSE index (sample January 2000 to December 2015), the skewness is negatively skewed and also exist negative kurtosis which indicate anomalous distribution. Shapiro-Wilk test indicate non-normality. While Figure 1 is the time plot pointing toward non-stationary series and Figures 2 and 3 are the Autocorrelation Function (ACF) and Partial Autocorrelation Function (PACF) which pointed towards AR(1) model for the data.

II. RESULTS

Table 1: Descriptive statistics for Returns

INDEX	**MIN**	**MEDIAN**	**MEAN**
NSE	1.00	96.50	96.40
MAX	**SKEWNESS**	**KURTOSIS**	**SHAPIRO-WILK TEST**
191.00	-0.00763	-1.2260	0.9538

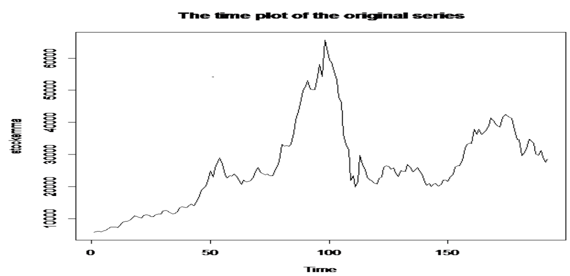

Figure 1: Time plot of the Series

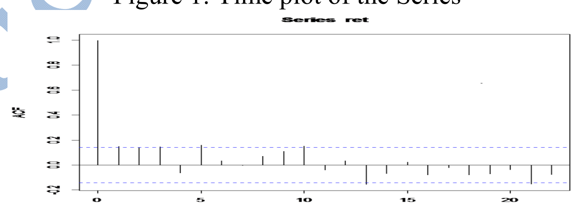

Figure 2: Autocorrelation function of Returns

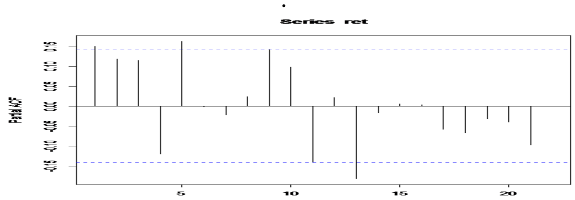

Figure 3: Partial Autocorrelation Function of Returns

III. DISCUSSIONS

A. *Choosing Volatility Model*

The basic estimation model consist of two equations, one for the mean which is a simple autoregressive AR(1) model and another for the variance which is identified by a particular ARCH specification i.e. GARCH (1,1). For NSI, the models are estimated using R code by the approximate quasi- maximum likelihood estimator assuming normal, student t, GED and Generalized Beta Skewed-t as innovations. To compare the different densities with model we apply the Akaike Information Criterion (AIC) and log likelihood values. When we analyze the densities we find that beta skewed-t distribution clearly out performed the normal distribution. Indeed the log likelihood function increases when using beta skewed-t distribution, leading to AIC criteria of 10.002 and 10.413 for normal density.

Table II: Table of results of model selection criteria between the existing and new proposed innovations.

GARCH TYPE	LOGLIKELIHOOD	AIC
GARCH-NORM	-995.69	10.41341
GARCH-STD	-1002.17	10.49134
GARCH-GED	-953.46	10.983940
GARCH-BST	-1040.15	10.00341

B. Forecasting

The forecasting we obtain are evaluated using Root Mean Square Error (RMSE) and Mean Absolute Percentage Error (MAPE) predicting 24 steps ahead. The forecasting is reported by ranking the different models with respect to RMSE and MAPE for NSE index, the result support the asymmetric GARCH than the symmetric GARCH model The Generalized beta skewed-t- GARCH(1,1) is the most successfully in forecasting NSE conditional variance.

II. Conclusion

The Model performance of GARCH (1,1) model was compared using different distributions for Nigeria Stock Index returns. We found that the new proposed GARCH(1,1) – Beta Skewed-t model is the most promising for characterizing the dynamic behaviour of these returns as it reflects their underlying process in terms of serial correlation, asymmetric volatility clustering and anomalous innovation. We recommend that researchers, Portfolio users consider GARCH (1,1) – Beta Skewed-t model in modeling Nigeria Stocks Market as it has proven better to the convectional ones.

Table III: Forecasting Analysis for the NSE Index: Comparing between densities

MODEL	RMSE	MAPE
GARCH-NORM	0.7565	0.7565
GARCH- STD	0.3899	0.3421
GARCH- GED	0.6261	0.6840
GARCH-BST	**0.3094***	**0.3005***

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