A new $\bar{X}$-Chart for Detecting Small, Medium, and Large Shift in the Process Mean

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***Abstract* — This article proposes a new $\bar{X}$-chart for controlling the process mean based on repetitive sampling scheme. This chart makes use of two pairs of upper and lower control limits to effectively detect the process mean shift away from the target value for small, medium and large shift. The control limits coefficients k_1 and k_2 were obtained through Monte-Carlo simulation experiment by minimizing the out-of-control average run length and at the same time maximizing the in-control average run length. The new chart returns to the conventional $\bar{X}$-chart when $k_1 = k_2$. The proposed control chart has a comparative advantage over the synthetic control chart, the exponentially moving average (EWMA) chart, and the joint $\bar{X}$-EWMA chart in that it can detect both small and large shift in mean while the others work well only when the shift in mean is small. A real-life example is presented.**

Keywords: *$\bar{X}$-chart, Repetitive sampling, Process mean, Monte-Carlo simulation, Average run length.*

I. Introduction

Quality is all of the features and characteristics of a product or service that satisfy customer or consumers need. This implies that the product or service that is of standard quality bring about good reputation to the production industry, minimizes loss of good will and enables global market penetration. It is on this basis that continuous improvement in the quality of a product or a service becomes indispensable to industries. This continuous improvement in quality can only be made possible if industrialists pay substantial attention to the process management from the raw material stage through the manufacturing process up till the finished product or service.

One most important segment of quality control is the use of control charts in monitoring the production process so as to attain, sustain and improve on the standard or of a product or a service.

Control charts such as the $\bar{X}$-bar, *EWMA* and *CUSUM* are very useful tools for the improvement in the quality of the products or services. These types of control charts provide the facility to monitor the process and quick indication when the process is going to be out-of-control. The Shewhart control chart otherwise known as the $\bar{X}$-chart is effective in detecting large scale shift in the process mean. However, this chart fails to detect an out-of-control when the process shift is either moderate or small. (Wu and Trevor,2000). The exponentially weighted moving average chart of Robert (1959) known as *EWMA*-chart and the cumulative sum chart also known as *CUSUM* chart are powerful tool for detecting small shift in the mean (See

Crowder, 1987, Lucas and Saccucci, 1990). However, they both become inferior to the $\bar{X}$-chart when there is moderate to large shift in the mean. Albin et al (1997) proposed a joint $\bar{X}$-$EWMA$ chart in which an out-of-control is signalled when the $\bar{X}$ sample is outside the control limit of the $\bar{X}$-chart or when a sample value of the statistic z is beyond the control limits of the $EWMA$ chart. The joint $\bar{X}$-$EWMA$ chart may improve the $EWMA$ chart for detecting large shift in the process mean but at the expense of reducing its effectiveness for small shifts in the process mean.

Wu and Trevor (2000) proposed the synthetic $\bar{X}$-chart for detecting small shift in the mean. This chart is very effective in detecting small shift in the mean. However, the detection power of the synthetic $\bar{X}$-chart reduces drastically when the shift in the mean becomes large. Furthermore, the implementation of the synthetic $\bar{X}$-chart is somehow cumbersome as it requires the combination of the $\bar{X}$-chart and the conforming run length chart in addition to some numerical iterations and computations to derive the control limits and implement the chart.

Often times, attempts made towards improving the performance of the $\bar{X}$-chart are geared towards combining the $\bar{X}$-chart itself with other charts and hence the resulting chart may not be able to simultaneously detect small and large shift in the mean at a stable run length.

In this article, we proposed an improved version of the $\bar{X}$-chart that is capable of simultaneously detecting small and large shift in the mean at a stable run length. The new $\bar{X}$-chart is developed with two pairs of control limits such that the control limits coefficients k_1 and k_2 (with $k_1 \geq k_2$) were obtained through Monte-Carlo simulation experiment by minimizing the out-of-control average run length and at the same time maximizing the in-control average run length. The new chart however returns to the conventional $\bar{X}$-chart when $k_1 = k_2$ making it capable of detecting small and large shift in the mean.

II. Background of the Shewart $\bar{X}$ −Chart

Let us consider a random sample of m subgroups each of size n from a normal distribution. Supposed that X_{ij} denotes the measurement from the j^{th} sample in the i^{th} subgroup then, the data set alongside estimates arising from the measurement of a given quality characteristics is presented in Table I.

Table I: Sample Measure of Quality Characteristics

	$j-Samples$						
$i-Subgroups$	X_1	X_2	X_3	…	X_n	$\bar{X}_i$	S_i
1	x_{11}	x_{12}	x_{13}	…	x_{1n}	$\bar{X}_1$	S_1
2	x_{21}	x_{22}	x_{23}	…	x_{2n}	$\bar{X}_2$	S_2
3	x_{31}	x_{32}	x_{33}	…	x_{3n}	$\bar{X}_3$	S_3
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
m	x_{m1}	x_{m2}	x_{m3}	…	x_{mn}	$\bar{X}_m$	S_m

The i^{th} subgroup mean is calculated using

$$\mu_i = \bar{X}_i = \frac{\sum_{j=1}^{n} x_{ij}}{n} \tag{1}$$

While i^{th} subgroup standard deviation is calculated using

$$S_i = \sqrt{\frac{\sum_{j=1}^{n}\left(x_{ij} - \bar{X}_i\right)^2}{n-1}} \tag{2}$$

The control chart parameters are estimated as

$$\mu = \bar{\bar{X}} = \frac{\sum_{i=1}^{m} \bar{X}_i}{m} \tag{3}$$

$$\sigma = \bar{S} = \frac{\sum_{i=1}^{m} S_i}{m} \tag{4}$$

The resulting control limits are given as

$$\left.\begin{aligned} UCL &= \mu + k\sigma \\ CL &= \mu \\ LCL &= \mu - k\sigma \end{aligned}\right\} \tag{5}$$

The Shewart $\bar{X}$-chart plots the μ_i in Equation (1) against the subgroup number m such that any μ_i outside the control limits in Equation (5) signifies an out-of-control point otherwise, the process is said to be in a state of statistical (stable) control.

III. The proposed $m\bar{X}$ −Chart

Following from Aslam et al. (2014), the repetitive sampling plays important role in lot acceptance sampling plans because it helps in determining the lot quality of an outgoing product or service. This attracting quality of repetitive sampling is incorporated in the new $\bar{X}$-chart with the hope of detecting the process out-of-control faster since the decision is postponed by collecting more data in form of repetitive sampling. This means that the decision (whether the process is in control or out of control) will be made based on a single sample when it is obvious however, resampling will be made if the decision is not obvious. We refer to the proposed chart as the modified $\bar{X}$-chart and denote it as $m\bar{X}$-chart.

3.1 The $m\bar{X}$-chart Procedure

We propose the following two step procedure for the $m\bar{X}$-chart having double control limits based on repetitive sampling:

Step I: Take a sample of size n for the i^{th} subgroup and calculate the sample mean $\bar{X}_i$ using Equation (1)

Step II: Declare the process as out-of-control if $\bar{X}_i \geq UCL_1$ or $\bar{X}_i \leq LCL_1$. Declare in-control if $LCL_2 \leq \bar{X}_i \leq UCL_2$. Otherwise, go to Step I for resampling.

The operational procedure of the proposed chart is based on a pair of lower control limits denoted as $LCL_1$and LCL_2 and another pair of upper control limits denoted as UCL_1 and UCL_2 presented below.

$$\left.\begin{array}{ll} UCL_1 = \mu_{\bar{x}} + k_1\sigma, & LCL_1 = \mu_{\bar{x}} - k_1\sigma \\ UCL_2 = \mu_{\bar{x}} + k_2\sigma, & LCL_2 = \mu_{\bar{x}} - k_2\sigma \end{array}\right\} \quad (6)$$

In Equation (6), $k_1$and k_2 are the control limits coefficients such that $k_1 \geq k_2$ are to be determined. Whenever $k_1 = k_2$ then the proposed $m\bar{X}$ control chart becomes the Shewart $\bar{X}$ control chart and hence, the $m\bar{X}$ control chart is an improved version of the Shewart $\bar{X}$ control chart that can simultaneously detect small and large shift in the process mean.

The two control limit coefficients are introduced to regulate the average run length (ARL) and since it is not possible to obtain the theoretical derivations for the ARL, we used the Monte Carlo simulation to determine the control limit coefficients of the proposed $m\bar{X}$-chart. The values of k_1 and k_2 will be determined for various values of n. The complete Monte Carlo simulation procedure is described below.

3.2 Monte Carlo Algorithm for Determining the Control Chart Coefficients k_1 and k_2 for the Proposed $m\bar{X}$ −Chart

1. Evaluating the In-control $ARL_\delta(\delta = 0)$ to obtain k_1 and k_2

1.1. Select the sample size n
1.2. Select initial values of k_1 and k_2
1.3. Generate n random variables denoted as $X \sim N(\mu_0 = 0, \sigma^2 = 1)$
1.4. Compute the subgroup mean $\bar{X}_i$ and standard deviation S_i for the i^{th} subgroup
1.5. Follow the procedure of the proposed control chart and check if the process is declared as out-of-control. If the process is declared as in-control, go to Step 1.3. If the process is declared as out-of-control, record the number of subgroups so far as the in-control run length.
1.6. Repeat Steps 1.3 through 1.5 a sufficient number (10,000 say) of times to calculate the in-control ARL. If the in-control ARL is equal to the specified ARL0, then go to Step 2 with the current values of k_1 and k_2. Otherwise, modify the values of k_1 and $k_2$and repeat steps 1.3 to 1.6.

2. Evaluating the Out-of-control $ARL_\delta(\delta \neq 0)$

2.1. Generate n random variables denoted as $X_\delta \sim N(\mu = [\mu_0 + \delta], \sigma^2 = 1)$
2.2. Compute the subgroup mean $\bar{X}_{\delta i}$ and standard deviation $S_{\delta i}$ for the i^{th} subgroup
2.3. Follow the procedure of the proposed control chart and repeat 2.1 and 2.2 until the process is declared as out-of-control. Record the number of subgroups as an out-of-control run length.
2.4. Repeat step 2.1 to 2.3 a sufficient number (say 10,000) of times to obtain the out-of-control $\boldsymbol{ARL_\delta}$.

3.3 The $m\bar{X}$-chart Coefficients

Following the proposed procedure of the $m\bar{X}$-chart and the algorithm for determining k_1 and k_2, the values of the control coefficients for which $ARL_{(\delta=0)}$ is set at 370 is presented in Table II.

From Table II, we note that there is a specific trend in that as the sample size increase, the control limit coefficient decreases from 2.02 to 0.22 and hence, as $2 \leq n \leq 31$, the control chart coefficients are in the region $2.02 \leq k_1 \leq 1.43$ and $0.44 \leq k_2 \leq 0.22$.

3.4 The Evaluation of the Proposed $m\bar{X}$-chart

The performance of $m\bar{X}$-chart is evaluated by implementing the second part of the algorithm in Section 3.2 for values of δ in the region$0.5 \leq \delta \leq 2.0$. The out-of-control average run length $ARL_\delta(\delta = 0.5 \leq \delta \leq 2.0)$ for the proposed method is computed and compared with other control chart methods such as the $\bar{X}$ chart, EWMA chart, the joint $\bar{X} - EWMA$ chart, and the synthetic chart. Results obtained is presented in Figure I. From the ARL plot, it can be seen that:

i. For the five (5) control chart methods considered, the ARL is large when there is little or no shift in the process mean but the ARL reduced drastically as the process mean shift increases.

ii. The $\bar{X}$ chart and the joint $\bar{X} - EWMA$ chart has the largest ARL when there is little or no shift in the process mean shift and also maintained this ARL size even when the process mean shifted drastically when compared to the other charting methods.

iii. The proposed $m\bar{X}$ chart has a large ARL when there is little or no shift in the process mean. However, the ARL reduced drastically as the process mean shift increases making it suitable to control both small and large shift in the process mean.

iv. The performance of the synthetic chart and the $EWMA$ control chart methods are quite similar as they work well for small shift in the process mean but fails when the process mean shift becomes large.

v. In the overall, the proposed $m\bar{X}$ chart is able to compete favourably and outperformed the synthetic and the joint $\bar{X} - EWMA$ chart.

3.4 Numerical Illustration of the Proposed $m\bar{X}$-chart

The data for this illustration is taken from Montgomery (2009). The data describes the fill volume of a soft-drink beverage bottle measured (approximately) by placing a gauge over the crown and comparing the height of the liquid in the neck of the bottle against a coded scale. On this scale, a reading of zero corresponds to the correct fill height. The fifteen subgroups of size $n = 10$ is presented in Table III. Since the performance of $EWMA$, synthetic and the joint $\bar{X} - EWMA$ chart are quite similar, the proposed $m\bar{X}$ −chart is compared with the $\bar{X}$ and the $EWMA$ charts using the data.

From Figures 2, 3, and 4, it can be seen that while the $\bar{X}$-chart sentenced the process as being in a state-of-control, the $EWMA$-chart and the proposed $m\bar{X}$-chart are able to identify the out-of-control points and hence declared the process as out-of-control.

Table II: k_1 and k_2 values for the proposed $m\bar{X}$-chart at $ARL_{(\delta=0)} \approx 370$

$ARL_{(\delta=0)} \approx 370$			$ARL_{(\delta=0)} \approx 370$			$ARL_{(\delta=0)} \approx 370$		
n	k_1	k_2	n	k_1	k_2	n	k_1	k_2
2	2.02	0.47	12	1.57	0.27	22	1.52	0.26
3	2.03	0.44	13	1.58	0.28	23	1.52	0.26
4	2.03	0.39	14	1.58	0.28	24	1.53	0.26
5	1.84	0.38	15	1.57	0.28	25	1.53	0.26
6	1.83	0.38	16	1.56	0.27	26	1.52	0.25
7	1.71	0.37	17	1.56	0.27	27	1.49	0.24
8	1.65	0.31	18	1.53	0.26	28	1.50	0.23
9	1.63	0.30	19	1.54	0.26	29	1.48	0.23
10	1.60	0.27	20	1.55	0.26	30	1.46	0.22
11	1.60	0.28	21	1.54	0.27	31	1.43	0.22

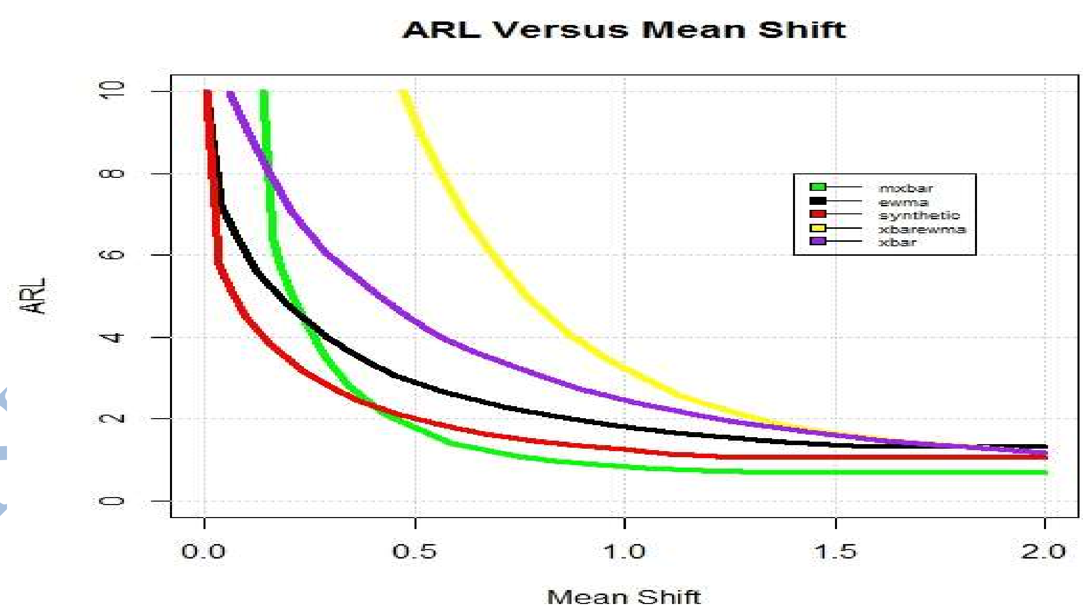

Figure 1: Average Run Length for Simulation Experiment

Table III : Montgomery (2009) Data

Sample	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
1	2.5	0.5	2	−1	1	−1	0.5	1.5	0.5	−1.5
2	0	0	0.5	1	1.5	1	−1	1	1.5	−1
3	1.5	1	1	−1	0	−1.5	−1	−1	1	−1
4	0	0.5	−2	0	−1	1.5	−1.5	0	−2	−1.5
5	0	0	0	−0.5	0.5	1	−0.5	−0.5	0	0
6	1	−0.5	0	0	0	0.5	−1	1	−2	1
7	1	−1	−1	−1	0	1.5	0	1	0	0
8	0	−1.5	−0.5	1.5	0	0	0	−1	0.5	−0.5
9	−2	−1.5	1.5	1.5	0	0	0.5	1	0	1
10	−0.5	3.5	0	−1	−1.5	−1.5	−1	−1	1	0.5
11	0	1.5	0	0	2	−1.5	0.5	−0.5	2	−1
12	0	−2	−0.5	0	−0.5	2	1.5	0	0.5	−1
13	−1	−0.5	−0.5	−1	0	0.5	0.5	−1.5	−1	−1
14	0.5	1	−1	−0.5	−2	−1	−1.5	0	1.5	1.5
15	1	0	1.5	1.5	1	−1	0	1	−2	−1.5

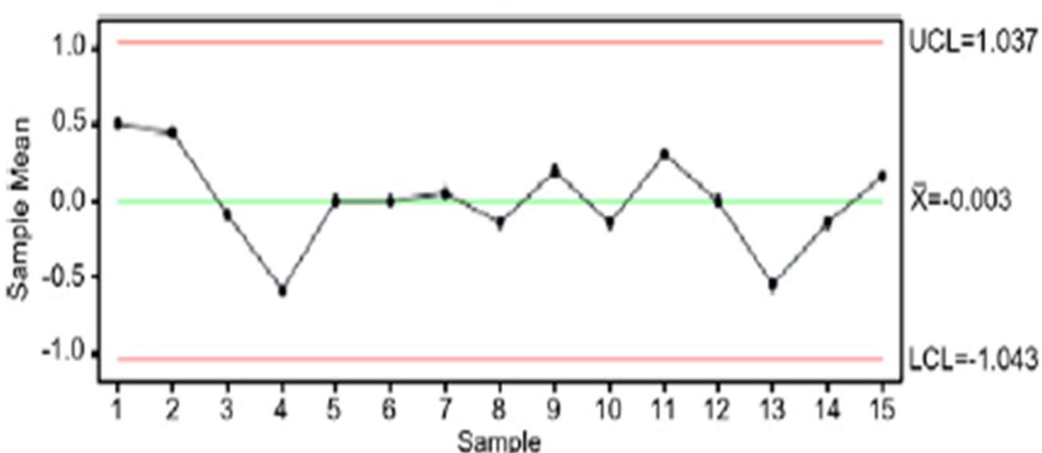

Figure 2: The Shewart X-bar Chart for Montgomery Data

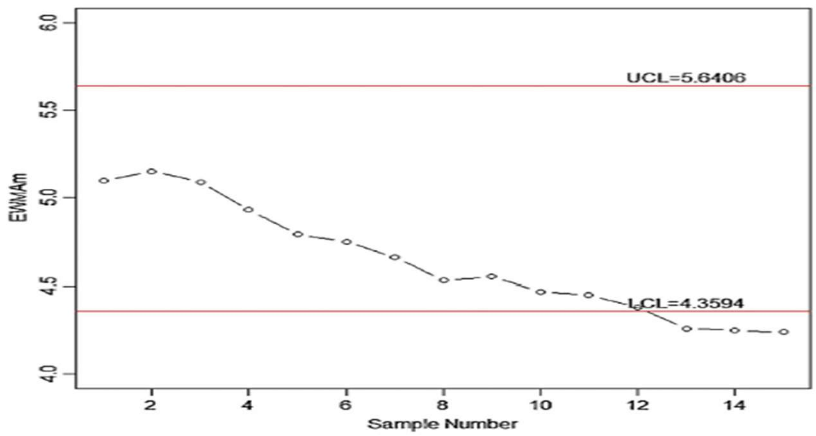

Figure 3: EWMA Chart for Montgomery Data

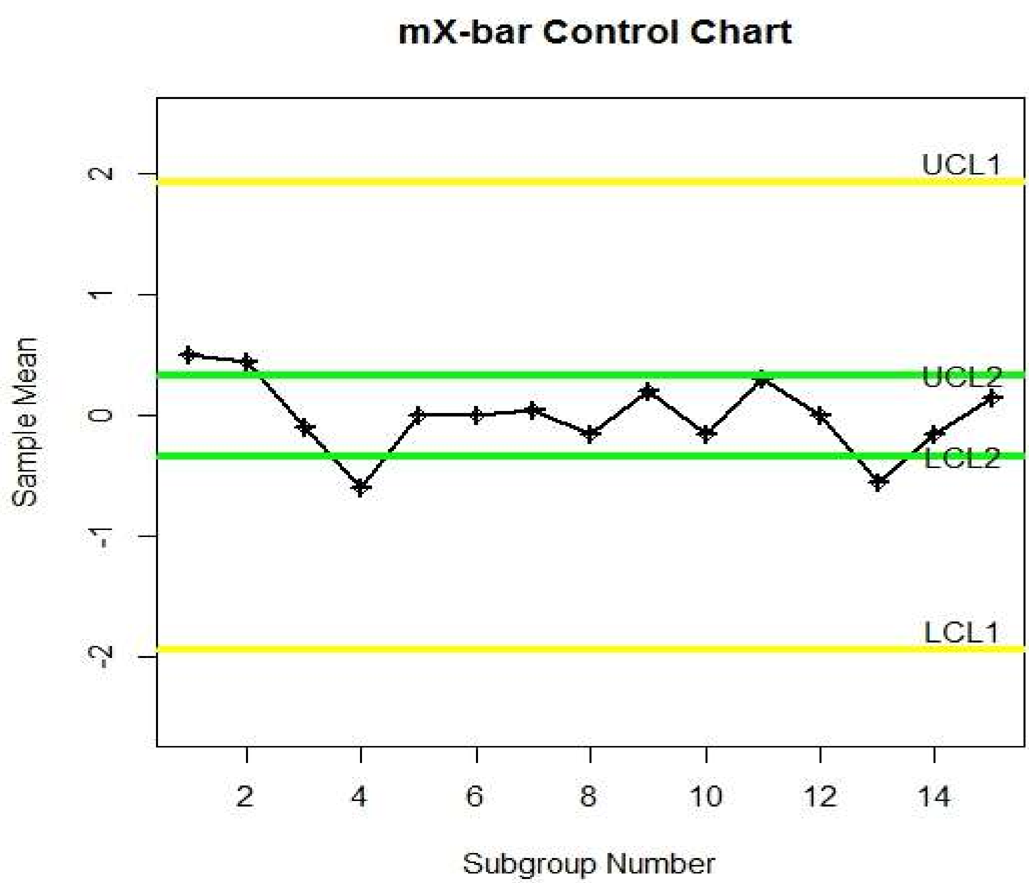

Figure 4: The mx-bar Chart for Montgomery Data

IV. Conclusion

This article proposes a control chart that is a modification of the Shewart $\bar{X}$-chart for simultaneously monitoring small and large shift in the process mean. The performance of the proposed chart is compared with existing charts such as the $EWMA$, the joint $\bar{X} - EWMA$, and the synthetic control chart. The proposed $m\bar{X}$-chart performed better that these charting techniques in that it can simultaneously control small and large shift in the process mean.

In summary, no single control chart method seems to be outstanding and for any given control chart technique, it is easier to find a process mean shift configuration weherein a method fails and hence, we state that the proposed $m\bar{X}$-chart may exhibit false alarm as the process mean shift increases.

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