Transmuted Weighted Weibull Distribution: Theory and Application

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***Abstract* — There are many ways of introducing additional parameter to any existing distribution in which one of the ways is quadratic rank transmutation method. Here, we introduce a transmuted parameter to two parameter weighted Weibull distribution called the transmuted weighted Weibull distribution to have a transform and flexible distribution including its special cases of distribution. Some of the properties of the propose distribution including: the reliability, hazard, reverse hazard function were obtained. We also provide the moment generating function, mean, variance and estimate the model parameter through the method of maximum likelihood estimation. The usefulness of the propose distribution is illustrated with a life data set of tensile fatigue characteristics of yarn and the results were compared with some other competing distributions with model selection criteria. Eventually, the result shown that the propose distribution perform better than other competing distributions.**

Keywords - *Moment, Reliability, Reverse Hazard, Quadratic Rank, Yarn.*

I. Introduction

There are different ways in which a parameter or more can be introduced to any existing distribution to transform the distribution. One of the ways is quadratic transmutation method. Therefore, we introduce a transmuted parameter (λ) to an existing weighted Weibull distribution by Shahbaz *et al*. 2010 called Transmuted Weighted Weibull (TWW) distribution. Numerous researchers have worked on weighted Weibull distribution in different ways including the authors of this article. None of these researchers have not use the distribution with quadratic transmutation. Although, some have worked on Transmuted Weibull, Exponential, extreme value, Quasi Lindley, beta Weibull such as Gokarna and Chris 2009, 2011, Elbatal and Elgarhy 2013, Manisha and Montip 2014, Lucena *et al*. 2015, Elgarhy *et al*. 2016 to mention but few.

II. The Transmuted Weighted Weibull (TWW) Distribution

$$A(m) = (1+\lambda)F(m) - \lambda F(m)^2, \ |\lambda| \le 1 \qquad (1)$$

The cdf of the WW distribution is defined as

$$F(m) = \frac{k+1}{k}\left[\left(1-\exp\left(-m^{\beta}\right)\right) - \frac{1}{k+1}\left(1-\exp(-(1+k)m^{\beta})\right)\right] \qquad (2)$$

and the pdf is

$$f(m) = \beta m^{\beta-1}\exp\left(-\left(m^{\beta}\right)\right)\left[1-\exp(-(km^{\beta}))\right] \qquad (3)$$

Putting (1 and 2) together which result to (3), we then have cdf called transmuted weighted Weibull (TWW) distribution and is expressed as

$$F(m) = \frac{k+1}{k}\left[\left(1-\exp\left(-m^{\beta}\right)\right) - \frac{1}{k+1}\left(1-\exp(-(1+k)m^{\beta})\right)\right]\left[1+\lambda exp(-(1+k)m^{\beta})\right] \qquad (4)$$

and the associative pdf of the TWW distribution also defined by

$$f(m) = \beta m^{\beta-1}\exp\left(-\left(m^{\beta}\right)\right)\left[1-\exp(-(km^{\beta}))\right]\left[1+\lambda exp(-(1+k)m^{\beta})\right] \qquad (5)$$

Furthermore, some of the properties of the propose distribution were obtained such as: reliability, Hazard, reverse hazard, moment generating function, mean and variance. However, the properties were obtain as follows:

(a) The Reliability Function (RF) of the TWW distribution

$$RF(m) = 1 - I_{F(m,k,\beta,\lambda)}$$

where, $I_{F(m,\gamma,\beta,\lambda)}$ is in (4) above

$$RF(m) = 1 + k - \exp(-(km^{\beta}) \left[1 + \lambda exp(-(1 + k)m^{\beta}))\right] \quad (6)$$

(b) The Hazard function (HF) is (5) divided (6), we get

$$HF(m;\ k,\beta,\lambda) = \frac{f(m;\ k,\beta,\lambda)}{RF(m;\ k,\beta,\lambda)}$$

$$= \frac{\beta(1+k)m^{\beta-1}(1-\exp(-km^{\beta}))[(1-\lambda+2\lambda)\exp(-km^{\beta})]}{1+k-\exp(-km^{\beta})[1+\lambda\exp(-(1+k)m^{\beta})]} \quad (7)$$

(c) The Reverse Hazard Function (RHF) is also (5) divided (4) above which gives

$$HRF(m;\ k,\beta,\lambda) = \frac{f(m;\ k,\beta,\lambda)}{F(m;\ k,\beta,\lambda)}$$

$$HRF(m;\ \gamma,\beta,\lambda) = \frac{\beta m^{\beta-1}\exp(-m^{\beta})(1-\exp(-km^{\beta}))[(1-\lambda+2\lambda)\exp(-km^{\beta})]}{\left[1-\exp(-m^{\beta})-\frac{1}{k+1}(1-\exp(-(1+k)m^{\beta}))[1+\lambda\exp(-(1+k)m^{\beta})]\right]} \quad (8)$$

(d) The Generating Function (GF) of the pdf in (5) is defined as

$$GF(m;\ k,\beta,\lambda) = \int_{0}^{\infty} e^{tm} f(m;\ k,\beta,\lambda)\ dm$$

$$= \int_{0}^{\infty} e^{tm}\frac{k+1}{k}\beta m^{\beta-1}e^{-m^{\beta}}\left(\frac{1}{-\exp(-km^{\beta})}\right)[(1-\lambda + 2\lambda)\exp(-km^{\beta})]dm$$

$$E(M^{r}) = \sum \frac{m!}{k!}\left[1 - k - (1-k)\frac{i}{\beta}\right]\Gamma\left((1-\lambda+2\lambda) + \frac{i}{\beta}\right)$$

and the rth moment is written from immediate above expression

$$\mu_r' = \frac{1}{k}\left[1 + k - (1+k)\frac{r}{\beta}\right]\ \Gamma\left((1-\lambda+2\lambda) + \frac{r}{\beta}\right) \quad (9)$$

Then, suppose $r = 1, 2 \ldots$ we have the mean and the variance as

$$\mu_1' = \frac{1}{k}\left[1 + k - (1+\gamma)\frac{1}{\beta}\right]\ \Gamma\left((1-\lambda+2\lambda) + \frac{1}{\beta}\right) \quad (10)$$

Also, for r = 2, we obtain the variance as

$$\mu_2' = \frac{1}{\gamma}\left[1 + k - (1+k)\frac{2}{\beta}\right]\Gamma\left((1-\lambda+2\lambda) + \frac{2}{\beta}\right) \quad (11)$$

It must be noted that (10) and (11) are both first-second moment about the mean of the TWW Distribution.

III. Estimation of Parameter

The pdf of TWW distribution considering in (5) with the likelihood function for a sample size n is given as: Suppose $x_1, \ldots, x_n$ are sample size n from TWW distribution. Therefore, the likelihood function n is stated below:

$$LF(m;\ k,\beta,\lambda) = \left(\frac{k+1}{k}\right)^{n}\beta^{n}\prod_{i=1}^{n} m_i^{\beta}\, exp\left[-\sum_{i=1}^{n} m_i^{\beta}\prod_{i=1}^{n}(1 - \exp(-km_i^{\beta}))\right]\left[(1-\lambda+2\lambda)\exp(-km_i^{\beta})\right] \quad (12)$$

Then, by taking the logarithm of (12) above, we obtain (13) below

$$\text{InL} = \text{nIn}\left(\frac{k+1}{k}\right) + nln\beta + \sum_{i=1}^{n}(\beta-1)lnm_i - \sum_{i=1}^{n} m_i^{\beta} + \sum_{i=1}^{n} ln(1-\exp(-km_i^{\beta})) + \sum_{i=1}^{n}\ln((1-\lambda+2\lambda)\exp(-km_i^{\beta})) \quad (13)$$

Through the study, we also obtained the partial derivatives of (12) concerning k, β and λ to satisfy the following normal equations.

$$\frac{\partial lnLn}{\partial k} = -\frac{n}{k} + \frac{n}{k+1}\sum_{i=1}^{n}\frac{m_i^{\beta}\exp(-km_i^{\beta})}{1-\exp(-km_i^{\beta})} - \sum_{i=1}^{n} 2\lambda\sum_{i=1}^{n}\frac{klnm_i^{\beta}\exp(-km_i^{\beta})}{[((1-\lambda+2\lambda)\exp(-km_i^{\beta}))]} = 0 \quad (14)$$

With parameters $(\alpha_2 + \frac{\psi_1}{2} + n - 1, \beta_1 + \sum x + \frac{1}{2})$

$$\frac{\partial lnLn}{\partial \beta} = -\frac{n}{\beta} + \sum_{i=1}^{n} lnm_i - \sum_{i=1}^{n} x_i^{\beta} lnm_i + \sum_{i=1}^{n}\frac{klnm_i^{\beta}\exp(-km_i^{\beta})}{1-\exp(-km_i^{\beta})} - \sum_{i=1}^{n}\frac{\beta lnm_i^{\beta}\exp(-km_i)}{[((1-\lambda+2\lambda)\exp(-km_i^{\beta}))]} = 0 \quad (15)$$

$$\frac{\partial lnLn}{\partial \lambda} = \sum_{i=1}^{n}\frac{2\exp(-km_i^{\beta})-1}{[(1-\lambda+2\lambda\exp(-km_i^{\beta}))]} = 0 \quad (16)$$

However, non-linear system of equations can be obtained by estimating the ML estimator $\hat{\alpha} = (\hat{k}, \hat{\beta}, \hat{\lambda})$ of $\alpha = (k, \beta, \gamma)$ using Newton Ralphson algorithm to maximize the LL in equation (13) above.

Hence, we may compute both standard error and asymmetric confidence interval following Aryal and Tsokos (2011) as n tends to infinity, then asymptotic distribution of the MLE $(\hat{k}, \hat{\beta}, \hat{\lambda})$ which is written in (17) below

$$\begin{pmatrix}\hat{K}\\ \hat{\beta}\\ \hat{\lambda}\end{pmatrix} \sim N\left[\begin{pmatrix}K\\ \beta\\ \lambda\end{pmatrix}, \begin{pmatrix}\hat{P}_{11} & \hat{P}_{12} & \hat{P}_{13}\\ \hat{P}_{21} & \hat{P}_{22} & \hat{P}_{23}\\ \hat{P}_{31} & \hat{P}_{32} & \hat{P}_{33}\end{pmatrix}\right] \quad (17)$$

where, $\hat{P}_{ij} = P_{ij}|_{\alpha=\hat{\alpha}}$ and

$$\begin{pmatrix}P_{11} & P_{12} & P_{13}\\ P_{21} & P_{22} & P_{23}\\ P_{31} & P_{32} & P_{33}\end{pmatrix} = \begin{pmatrix}V_{11} & V_{12} & V_{13}\\ V_{21} & V_{22} & V_{23}\\ V_{31} & V_{32} & V_{33}\end{pmatrix}^{-1}$$

The above matrix is called variance-covariance matrix and its elements are as follows:

$$V_{11} = -\frac{\partial^2 L_n}{\partial \gamma^2},\ V_{12} = -\frac{\partial^2 L_n}{\partial\gamma\partial\beta},\ V_{22} = -\frac{\partial^2 L_n}{\partial\beta^2},$$

$$V_{23} = -\frac{\partial^2 L_n}{\partial\gamma\partial\lambda},\ V_{33} = -\frac{\partial^2 L_n}{\partial\lambda^2},\ V_{13} = -\frac{\partial^2 L_n}{\partial\beta\partial\lambda}$$

An approximation $(1-\alpha)100\%$ having two sides C.Is for k, β and λ were respectively and stated as

$\hat{k} \pm Z\alpha_{/2}\sqrt{\hat{A}_{11}}$, $\hat{\beta} \pm Z\alpha_{/2}\sqrt{\hat{A}_{22}}$ and $\hat{\lambda} \pm Z\alpha_{/2}\sqrt{\hat{A}_{33}}$

IV. The Order Statistics

Here, we studied and obtained the $yth, nth, first$ and when $n = 2$ order statistics of the WW distribution as follows: Suppose $M, \ldots, M_{(n)}$ is order statistics having a random sample $m_1, \ldots, m_n$ out of a population with p.d.f $f_M(m)$ and c.d.f $F_M(m)$ then, the p.d.f of M_y is

$$f_{M_y}(m) = \frac{n!}{(y-1)!(n-y)!} f_M(m)[F_M(m)]^{y-1}[1-F_M(m)]^{n-y} \quad (18)$$

$m = 1, \ldots, n.$

By substituting (2) and (3) into (18) we have the p.d.f of the yth order WW random variable $M_{(y)}$ is

$$f_M(m) = \frac{n!}{(y-1)!\,(n-y)!}\frac{k+1}{k}\beta m^{\beta-1}\exp(-m^\beta)$$
$$\left(1-\exp(-(n+1-y)km^\beta)\right)1-1-\exp\left(-m^\beta\right)$$
$$\left[-\frac{1}{k+1}\left(1-exp\left(-(1+k)\right)\right)m^\beta\right]^{y-1}$$

Thereafter, the p.d.f of nth order WW statistics $M_{(n)}$ is

$$f_{M_y}(m) = \frac{n(k+1)}{k}\beta m^{\beta-1}\exp(-m^\beta)\left(1-\exp(-km^\beta)\right)$$
$$\left[1-\exp\left(-m^\beta\right)-\frac{1}{k+1}\left(1-exp\left(-(1+k)\right)\right)m^\beta\right]^{n-1} \quad (19)$$

and the p.d.f of the 1st order WW statistic $M_{(1)}$ is expressed below:

$$f_{M_1}(m) = \frac{n(k+1)}{k}\beta m^{\beta-1}\exp(-m^\beta)\left(1-\exp(-nkm^\beta)\right) \quad (20)$$

Then, in a case when $n = 2$ in (19) becomes

$$f_{M_2}(m) = \frac{2(k+1)}{k}\beta m^{\beta-1}\exp(-m^\beta)\left(1-\exp(-km^\beta)\right)$$
$$\left[1-\exp\left(-m^\beta\right)-\frac{1}{k+1}\left(1-exp\left(-(1+k)\right)\right)m^\beta\right] \quad (21)$$

and (20) also can be written as

$$f_{M_2}(m) = \frac{2(k+1)}{k}\beta m^{\beta-1}\exp(-m^\beta)\left(1-\exp(-2km^\beta)\right)$$
(22)

Meanwhile, (21) and (22) are sub-special cases of (5) for λ minus 1 and λ equal to 1 respectively. Furthermore, p.d.f of the kth order statistic for the TWW distribution is

$$f_{M_{(y)}}(m) = \frac{n!}{(y-1)!\,(n-y)!}\frac{k+1}{k}\beta m^{-1}\exp(-m)$$
$$[1-\exp(-km)][1-\lambda+2\lambda\exp(-km)][1+\lambda\exp(-(1$$
$$(+k)q)]^{y-1}\left[1-\exp(-q)-\frac{1}{k+1}(1-\exp(-(1+k)q))\right]^{y-1}$$
$$\exp[-(n-y)kq[(1-\lambda+\lambda)\exp(kq)]]$$

where, $q = m^\beta$. Hence, the p.d.f of the highest order statistic $M_{(n)}$ is expressed as

$$f_{M_{(n)}}(m) = \frac{n(k+1)}{k}\beta m^{\beta-1}\exp\left(-m^\beta\right)\left[1-\exp\left(-km^\beta\right)\right]$$
$$\left[(1-\lambda+2\lambda)\exp\left(-km^\beta\right)\right]\left[(1+\lambda)\exp\left(-km^\beta\right)\right]^{n-1}$$
$$\left[1-\exp\left(-m^\beta\right)-\frac{1}{k+1}\left(1-\exp\left(-(1+k)m^\beta\right)\right)\right]^{n-1}$$

In the same vein, the p.d.f of the least order statistic $M_{(1)}$ is written as:

$$f_{M_{(1)}}(m) = \frac{n(k+1)}{k}\beta m^{\beta-1}\exp\left(-m^\beta\right)\left[1-\exp\left(-km^\beta\right)\right]$$
$$\left[(1-\lambda+2\lambda)\exp\left(-km^\beta\right)\right].\left[(1+\lambda)\exp\left(-km^\beta\right)\right]$$
$$\left[(1-\lambda+\lambda)\exp\left(-m^\beta\right)-\frac{1}{k+1}\left(1-\exp\left(-(1+k)m^\beta\right)\right)\right]^{n-1}$$

Therefore, when $\lambda = 0$ the above expression becomes the order statistic of the two parameters weighted-Weibull distribution

V. Results and Discussion

A. Results

The summary of the data used is shown in Table 1 and the data were extracted from Manisha and Montip (2014). It is a set of data on failure time of 100centimetres yarn at 2.3% strain level with size $n = 100$. The estimate of the parameters and Standard errors in parentheses in each distribution including: Weighted Weibull (WW), Transmuted Weibull (TW) and Transmuted Weighted Weibull (TWW) distribution and their associating values such as Log-likelihood, Akaike Information Criterion, Bayesian Information Criterion and Consistent Akaike Information Criterion were gotten by the use of R software and its shown in Table 2 below.

The value of skewness and kurtosis shown in Table one clearly shows that the data is skewed in nature. The histogram plots of density curves and distribution of the set of data in figure two also testified to the nature of the data (i.e it's skewed to the left). Hence, any skew data requires more flexible distribution that can accommodate and adequate for its skewness; of which our proposed distribution (TWW) has proved among the generalised distributions studied as we have shown in the plots above (figure one and two).

Shape of Density, Distribution, Reliability, Hazard, and Reverse Hazard Functions

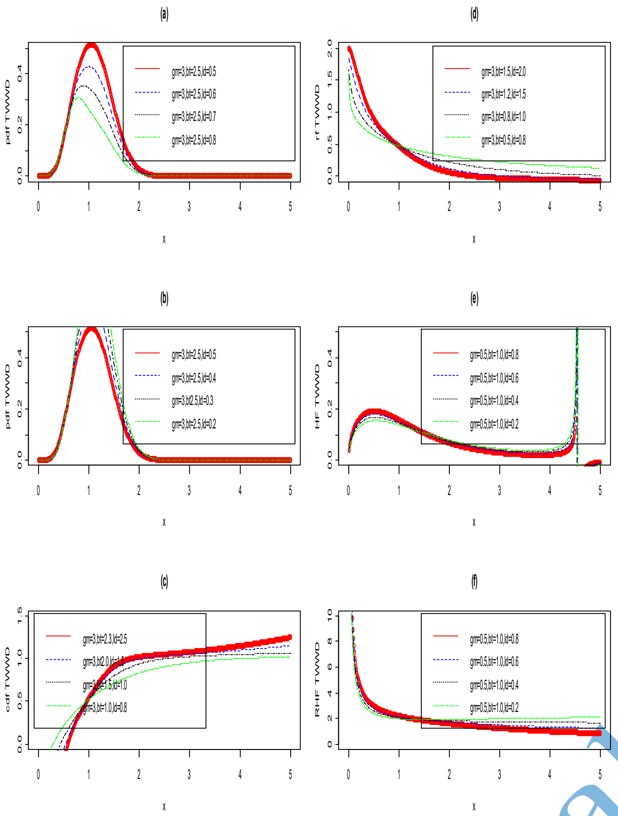

Figure 1: consists a – f plots above are the graphical representation of the pdf, cdf, RF, HF, and RHF of TWW distribution. a and b are pdf plots of the TWW distribution in (5) for different values of transmuted parameter (ld = lamda) while, shape (bt = beta) and scale (gm = gamma) parameters are constant. c is the cdf plot (4) is an increasing function, d is the RF in (6) with different values of beta and lamda parameters while gamma is constant, e and f are the plots of HF and RHF in (7 and 8) with different values of lamda while value of gamma and beta are constant.

Table 1: Summary of the Data

Min	1st Q	Med	Mean
15.000	127.500	195.000	214.700
3rd Q	Maxi	Sk	Kt
270.500	829.000	1.512	6.778

Table 2 below consists the following: The estimate of the parameters and SEs in parentheses in each distribution including: Weighted Weibull (WW), Transmuted Weibull (TW) and Transmuted Weighted Weibull (TWW) distribution and associating values of (LL, AIC, BIC and CAIC.

Distribution	γ	β	λ
WW	0.25 (2.42)	0.15 (7.47)	1.00 (0.00)
TW	0.25 (4.11)	1.00 (0.00)	0.05 (4.19)
TWW	0.15 (1.05)	0.40 (2.97)	0.05 (3.15)
LL	**AIC**	**BIC**	**CAIC**
-536.34	1078.69	1086.50	1087.50
-379.78	763.55	768.76	769.76
-263.40	**530.80**	**536.00**	**537.00**

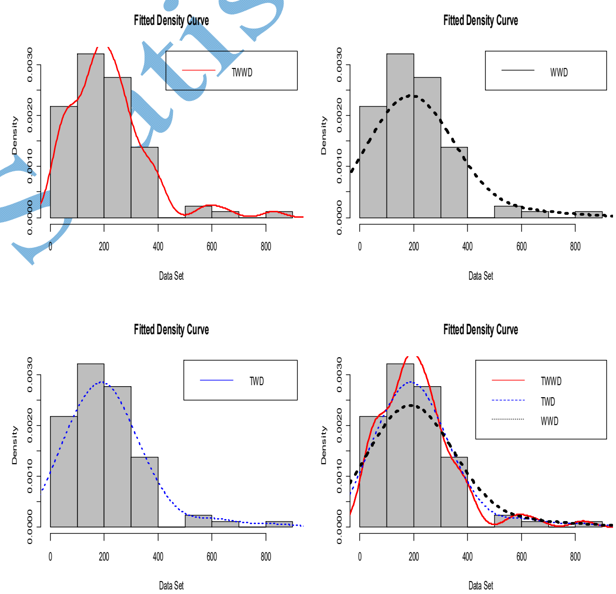

Figure 2: Plots of the fitted densities including; Transmuted Weighted Weibull (TWW), Transmuted Weibull (TW), Weighted Weibull (WW) and distribution of the data set.

The chain 1 is considered for convergence diagnostics plots. The visual summary is based on posterior sample obtained from chain 2 whereas the numerical summary is presented for both the chains.

B. Discussion

From Table 2 above, we have the estimated parameters and model selection criteria such as: LL, AIC, BIC and CAIC. Then, values of each of the model selection has shown the efficiency and flexibility of each distribution. For instance, the smaller the value of any model selection criteria of a distribution, the more efficient and flexible the distribution is. Therefore, our propose distribution (TWW) has proved its efficiency and flexibility over other competing distributions studied. However, it is better fitting and use to analyze real life phenomena

VI. Conclusion

In this research work, the researcher considered different distribution in estimating parameters and SEs in parentheses in each distribution including: Weighted Weibull (WW), Transmuted Weibull (TW) and Transmuted Weighted Weibull (TWW) distribution and associating values of (LL, AIC, BIC and CAIC.
Based on the results obtained, it was observed that our propose distribution (TWW) has proved its efficiency and flexibility over other competing distributions studied.

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