Theoretical Analysis of Exponentiated Transmuted Exponential Distribution

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***Abstract*—The two-parameter Transmuted Exponential Distribution (TED) was generalized to its three-parameter distribution entitled Exponentiated Transmuted Exponential Distribution (ETED). Mathematical expressions for its Moments, Moment Generating Function (MGF), Reliability analysis, Quantile Function and the Limiting behaviour of the proposed model were presented. The parameters of the proposed distribution were estimated using the method of maximum likelihood**

Keywords- *Exponentiated Transmuted Exponential Distribution, Moments, Moment generating function, Reliability analysis..*

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I. Introduction

The exponential distribution is one of the most important distribution with wide range of applications in practice. So many generalized models have been studied and widely used in statistics for modeling real datasets. This therefore, creates more rooms for proposing new distributions which could provide greater flexibility and wider acceptability in the modeling of real lifetime datasets.

Gupta and Kundu [1] first proposed a generalization of the standard exponential distribution, called the exponentiated exponential (EE) distribution. Owoloko et al. [2] studied the performance rating of the Transmuted Exponential distribution with regard to some other generalized models. The Transmuted Exponential distribution appeared to be better than the Beta Exponential distribution, Generalized Exponential distribution also known as Exponentiated Exponential distribution in terms of flexibility when applied to real life data. Oguntunde and Adejumo [3] proposed a new two parameter model called Transmuted Inverse Exponential Distributions by using a quadratic rank transmutation map and also derived some of its basic statistical properties. Nadarajah and Kotz [4] proposed, based on the same idea, four more exponentiated type distributions to extend and derive the structural properties of the standard gamma, standard Weibull, standard Gumbel and standard Fréchet distributions. Nadarajah [5] for Exponentiated Gumbel, Shirke and Kakade [6] for exponentiated log-normal and Nadarajah and Gupta [7] for exponentiated gamma distributions. The main aim of this paper is to increase the flexibility of the Transmuted Exponential Distribution there by making it to appropriately fit in data of various shapes that cannot be fitted with the existing once.

II. Materials and Methods

The Exponentiated family of distribution is derived by raising the cdf of an arbitrary parent distribution by a shape parameter say; $\alpha > 0$. Its pdf is given by;

$$F(x) = \left[G(x)\right]^{\alpha} \qquad (1)$$

Its corresponding probability density function (pdf) is given by;

$$f(x) = \alpha g(x)\left[G(x)\right]^{\alpha-1} \qquad (2)$$

The cdf and pdf of Transmuted exponential Distribution is given as;

$$G(x) = \left(1 - e^{-x/\theta}\right)\left(1 + \lambda e^{-x/\theta}\right)$$

$$g(x) = \frac{1}{\theta} e^{-x/\theta}\left(1 - \lambda + 2\lambda e^{-x/\theta}\right)$$

This shows that, the cdf and pdf of Exponentiated Transmuted Exponential Distribution are given as;

$$F(x) = \left[\left(1 - e^{-x/\theta}\right)\left(1 + \lambda e^{-x/\theta}\right)\right]^{\alpha} \qquad (3)$$

$$f(x)=\alpha\frac{1}{\theta}e^{-x/\theta}\left(1-\lambda+2\lambda e^{-x/\theta}\right)\left[\left(1-e^{-x/\theta}\right)\left(1+\lambda e^{-x/\theta}\right)\right]^{\alpha-1} \quad (4)$$

From equation (4);

$$\left(1-e^{-x/\theta}\right)^{\alpha-1}=\sum_{j=0}^{\infty}(-1)^j\frac{\Gamma\alpha}{j!\Gamma(\alpha-j)}e^{-jx/\theta}$$

$$\left(1+\lambda e^{-x/\theta}\right)^{\alpha-1}=\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma\alpha}{j!\Gamma(\alpha-k)}\lambda^k e^{-kx/\theta}$$

So equation (4) gives;

$$f(x)=\frac{\alpha}{\theta}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\frac{\lambda^k\Gamma\alpha\Gamma\alpha}{j!k!\Gamma(\alpha-j)\Gamma(\alpha-k)}\times e^{-\frac{(1+j+k)x}{\theta}}\left(1-\lambda+2\lambda e^{-x/\theta}\right)$$

Let, $m=\frac{\alpha}{\theta}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\frac{\lambda^k\Gamma\alpha\Gamma\alpha}{j!k!\Gamma(\alpha-j)\Gamma(\alpha-k)}$

Therefore, $f(x)=me^{-\frac{(1+j+k)x}{\theta}}\left(1-\lambda+2\lambda e^{-x/\theta}\right)$

By choosing some arbitrary values for parameters: we provide a possible shape for the CDF and pdf of the ETED as shown in Figures 1 and 2:

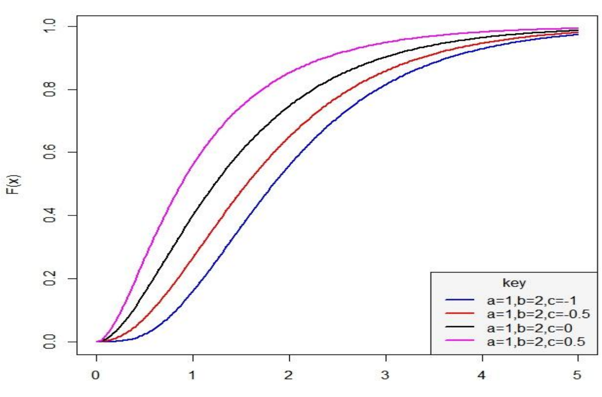

Figure 1: CDF of ETED

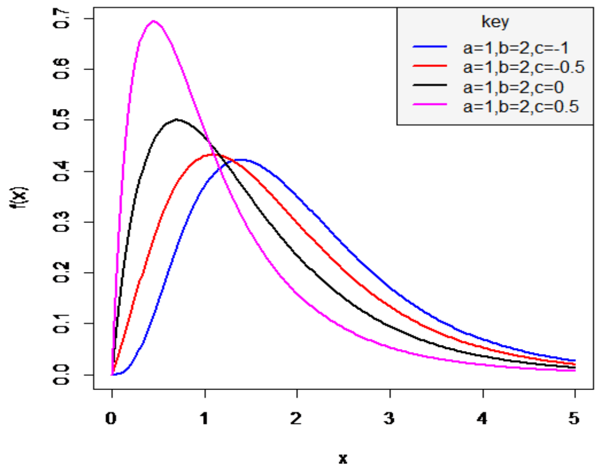

Figure 2: pdf of ETED

The rth Moment

Lemma 1: If X has the ETED distribution, then the rth moment of X is given as follows;

$$\mu_r'=\theta^r\Gamma(r+1)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\frac{\lambda^k\Gamma(\alpha+1)\Gamma\alpha}{j!k!\Gamma(\alpha-j)\Gamma(\alpha-k)}\times\left\{\frac{(1-\lambda)}{(1+j+k)^{r+1}}+\frac{2\lambda}{(2+j+k)^{r+1}}\right\} \quad (5)$$

Proof:

$$\mu_r'=E(X^r)=\int_0^{\infty}x^r f(x)dx$$

$$\mu_r'=m\int_0^{\infty}x^r e^{-\frac{(1+j+k)x}{\theta}}\left(1-\lambda+2\lambda e^{-x/\theta}\right)dx$$

$$\mu_r'=m\left\{(1-\lambda)\int_0^{\infty}x^r e^{-\frac{(1+j+k)x}{\theta}}dx+2\lambda\int_0^{\infty}x^r e^{-\frac{(2+j+k)x}{\theta}}dx\right\}$$

$$\mu_r' = m\left\{ \begin{aligned} &(1-\lambda)\int_0^\infty \left(\frac{\theta y}{1+j+k}\right)^r e^{-y}\frac{\theta dy}{1+j+k} \\ &+2\lambda\int_0^\infty \left(\frac{\theta y}{2+j+k}\right)^r e^{-y}\frac{\theta dy}{2+j+k} \end{aligned} \right\}$$

$$\mu_r' = m\left\{(1-\lambda)\frac{\theta^{r+1}\Gamma(r+1)}{(1+j+k)^{r+1}} + 2\lambda\frac{\theta^{r+1}\Gamma(r+1)}{(2+j+k)^{r+1}}\right\}$$

$$\mu_r' = \theta^r\Gamma(r+1)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\frac{\lambda^k\Gamma(\alpha+1)\Gamma\alpha}{j!k!\Gamma(\alpha-j)\Gamma(\alpha-k)}\times \left\{\frac{(1-\lambda)}{(1+j+k)^{r+1}}+\frac{2\lambda}{(2+j+k)^{r+1}}\right\}$$

Raw Moments about Origin

Putting r = 1, 2, 3 and 4 in (5) first four raw moments are;

$$\mu_1' = \theta\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\frac{\lambda^k\Gamma(\alpha+1)\Gamma\alpha}{j!k!\Gamma(\alpha-j)\Gamma(\alpha-k)}\times \left\{\frac{(1-\lambda)}{(1+j+k)^{2}}+\frac{2\lambda}{(2+j+k)^{2}}\right\}$$

$$\mu_2' = 2\theta^2\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\frac{\lambda^k\Gamma(\alpha+1)\Gamma\alpha}{j!k!\Gamma(\alpha-j)\Gamma(\alpha-k)}\times \left\{\frac{(1-\lambda)}{(1+j+k)^{3}}+\frac{2\lambda}{(2+j+k)^{3}}\right\}$$

$$\mu_3' = 6\theta^3\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\frac{\lambda^k\Gamma(\alpha+1)\Gamma\alpha}{j!k!\Gamma(\alpha-j)\Gamma(\alpha-k)}\times \left\{\frac{(1-\lambda)}{(1+j+k)^{4}}+\frac{2\lambda}{(2+j+k)^{4}}\right\}$$

$$\mu_4' = 24\theta^3\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\frac{\lambda^k\Gamma(\alpha+1)\Gamma\alpha}{j!k!\Gamma(\alpha-j)\Gamma(\alpha-k)}\times \left\{\frac{(1-\lambda)}{(1+j+k)^{5}}+\frac{2\lambda}{(2+j+k)^{5}}\right\}$$

The expressions for variance is;

$$\mathrm{var}(X) = E(X^2) - \left(E(X)\right)^2$$

$$\mathrm{var}(x) = 2\theta^2\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\frac{\lambda^k\Gamma(\alpha+1)\Gamma\alpha}{j!k!\Gamma(\alpha-j)\Gamma(\alpha-k)}\times \left\{\frac{(1-\lambda)}{(1+j+k)^{3}}+\frac{2\lambda}{(2+j+k)^{3}}\right\} - \left(\begin{aligned}&\theta\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\frac{\lambda^k\Gamma(\alpha+1)\Gamma\alpha}{j!k!\Gamma(\alpha-j)\Gamma(\alpha-k)}\times \\ &\left\{\frac{(1-\lambda)}{(1+j+k)^{2}}+\frac{2\lambda}{(2+j+k)^{2}}\right\}\end{aligned}\right)^2$$

The coefficient of variation can now be calculated using the following relationships

$$CV = \frac{\sqrt{\mathrm{var}(X)}}{E(X)}$$

Moment Generating Function

Lemma 2: If X has the ETED distribution, then the Moment Generating Function is given as follows;

$$M_x(t) = \frac{1}{\theta}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\frac{\lambda^k\Gamma(\alpha+1)\Gamma\alpha}{j!k!\Gamma(\alpha-j)\Gamma(\alpha-k)} \times\left\{(1-\lambda)\left(\frac{(1+j+k)}{\theta}-t\right)^{-1} + 2\lambda\left(\frac{(2+j+k)}{\theta}-t\right)^{-1}\right\} \quad (6)$$

Survival function

Survival function is the probability that a system will survive beyond a given time. Mathematically, the survival function is given by:

$$S(x) = 1 - F(x)$$

$$S(x) = 1 - \left[\left(1-e^{-x/\theta}\right)\left(1+\lambda e^{-x/\theta}\right)\right]^{\alpha} \quad (7)$$

Hazard function

Hazard function is also called the failure or risk function and is the probability that a component will fail or die for an interval of time. The hazard function is define as;

$$h(x) = \frac{f(x)}{S(x)}$$

$$h(x) = \frac{\alpha \frac{1}{\theta} e^{-x/\theta}\left(1-\lambda+2\lambda e^{-x/\theta}\right)\left[\left(1-e^{-x/\theta}\right)\left(1+\lambda e^{-x/\theta}\right)\right]^{\alpha-1}}{1-\left[\left(1-e^{-x/\theta}\right)\left(1+\lambda e^{-x/\theta}\right)\right]^{\alpha}} \quad (8)$$

Quantile Function

Lemma 3: If X has the Exponentiated transmuted Exponential Distribution, then the Quantile Function of X is given as follows;

$$x_u = -\theta \ln\left[1-\frac{(1+\lambda)\sqrt{(1+\lambda)^2-4\lambda u^{1/\alpha}}}{2\lambda}\right] \quad (9)$$

Limiting behavior

Here, we seek to investigate the asymptotic behavior of the ETED in equation (4) as $x \to 0$ and as $x \to \infty$

$$\lim_{x\to 0} f(x) = \lim_{x\to\infty} f(x) = 0$$

$$\lim_{x\to 0} f(x) = \alpha \frac{1}{\theta} e^{-x/\theta}\left(1-\lambda+2\lambda e^{-x/\theta}\right) \times \left[\left(1-e^{-x/\theta}\right)\left(1+\lambda e^{-x/\theta}\right)\right]^{\alpha-1} = 0$$

$$\lim_{x\to\infty} f(x) = \alpha \frac{1}{\theta} e^{-x/\theta}\left(1-\lambda+2\lambda e^{-x/\theta}\right) \times \left[\left(1-e^{-x/\theta}\right)\left(1+\lambda e^{-x/\theta}\right)\right]^{\alpha-1} = 0$$

The results of shows that the proposed model has at least a mode.

III. Estimation of Parameters of the ETED

The estimation of the parameters of the ETED is done by using the method of maximum likelihood estimation.

Let $X_1, X_2, \ldots\ldots, X_n$ be a random sample from the ETED with unknown parameter vector $\Theta = (\alpha, \theta, \lambda)^T$ .The total log-likelihood function for Θ is obtained from *f(x)* as follows;

$$\ln L(\Theta) = n \ln \alpha - n \ln \theta - \frac{\sum_{i=1}^{n} x_i}{\theta} + \sum_{i=1}^{n} \ln(1-\lambda+2\lambda e^{-x/\theta}) + \sum_{i=1}^{n} \ln\left[\left(1-e^{-x/\theta}\right)\left(1+\lambda e^{-x/\theta}\right)\right]^{\alpha-1}$$

$$\frac{\delta \ln L(\Theta)}{\delta \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln\left[\left(1-e^{-x/\theta}\right)\left(1+\lambda e^{-x/\theta}\right)\right] = 0$$

$$\frac{\delta \ln L(\Theta)}{\delta \lambda} \sum_{i=1}^{n} \frac{2e^{-x/\theta}-1}{(1-\lambda+2\lambda e^{-x/\theta})} + \sum_{i=1}^{n} \frac{(\alpha-1)\left[\left(1-e^{-x/\theta}\right)\left(1+\lambda e^{-x/\theta}\right)\right]^{\alpha-2}\left(e^{-x/\theta}+e^{-2x/\theta}\right)}{\left[\left(1-e^{-x/\theta}\right)\left(1+\lambda e^{-x/\theta}\right)\right]^{\alpha-1}} = 0$$

$$\frac{\delta \ln L(\Theta)}{\delta \theta} = \frac{n}{\theta} + \frac{\sum_{i=1}^{n} x_i}{\theta^2} - \sum_{i=1}^{n} \frac{2\lambda e^{-x/\theta}}{(1-\lambda+2\lambda e^{-x/\theta})} + \sum_{i=1}^{n} \frac{(\alpha-1)\left[\left(1-e^{-x/\theta}\right)\left(1+\lambda e^{-x/\theta}\right)\right]^{\alpha-2}\left(-\lambda/\theta\, e^{-x/\theta} + 1/\theta\, e^{-x/\theta} - 2\lambda/\theta\, e^{-x/\theta}\right)}{\left[\left(1-e^{-x/\theta}\right)\left(1+\lambda e^{-x/\theta}\right)\right]^{\alpha-1}} = 0$$

Hence, the MLE is obtained by solving this nonlinear system of equations. Solving this system of nonlinear equations is complicated, we can therefore use statistical software to solve the equations numerically.

IV. Conclusion

In this article, we introduces a new three parameter probability model called Exponentiated Transmuted Exponential Distribution. An additional shape parameters were added to the Transmuted Exponential Distribution in order to increase its flexibility. An explicit expression for some of its basic statistical properties were derived. Further research would involve applying the proposed model to real life data sets to evaluate its flexibility over the Transmuted Exponential Distribution.

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