Comparisons of Some Existing Exponential Cubic Rank Transmutation Distributions on Bladder Cancer Data

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***Abstract* — This study assesses the performances of five different Cubic Rank Transmutation (CRT) maps on the exponential distribution. Shapes of the hazard rate function, density function, and reliability functions are examined. Order statistics, raw moments, moment generating functions, and likelihood functions of each CRTE are obtained. Performances of the five distributions are examined using widely used lifetime data on bladder cancer patients. Despite having only 2 parameters, the CRTE distribution using Rahman *et al.* [10] transmutation map provides the best fit.**

Keywords: *Cubic Rank Transmutation Maps, exponential distribution, lifetime data*

I. Introduction

The generalization of baseline distributions is a common practice in probability theory. This is done with the objective of obtaining more flexible distributions that give a better result when modelling lifetime data. The process often involves the insertion of the density function of a baseline lifetime distribution like exponential, Weibull, Rayleigh, etc. into a generalized family of distribution. The resulting generalized distribution function is then analysed to obtain its different mathematical properties. Several forms of these generalization techniques pervade literature. The exponential family was developed by [1]; Beta G family of distributions was proposed by [2]; Marshall-Olkin family of distribution was introduced by [3]; Weibull G family was developed by [4]; the Kumaraswamy-G family is due to [5].

Since its introduction by [6], the Quadratic Rank Transmutation (QRT) map has been applied to various lifetime distributions. But due to its major deficiency in not being able to capture the complexity that always characterized data, it has been extended to Cubic Rant Transmutation (CRT) map by different authors [7, 8, 9, 10, 11]. Several lifetime distributions have been generalized using the CRT maps. Although the basis for all these maps is the QRT map of [6], there are some variations in the mathematical processes followed in obtaining each of them. This study is aimed at examining the flexibility and performance of five existing CRT maps using the exponential distribution as the baseline distribution. Some lifetime distributions that are generalized using the CRT map are provided in [12].

II. The Cubic Rank Transmutation Maps

Five CRT maps are examined on the exponential distribution in this study. The first one is the CRT due to [7]. Given a baseline distribution with *cdf* G(x), the CRT-family of distribution as:

$$F(x) = \alpha G(x) + (\beta - \alpha)\, G^2(x) + (1 - \beta)\, G^3(x) \qquad (2.1)$$

where $\alpha \in [0, 1]$ and $\beta \in [-1, 1]$. This method was used on Weibull and log-logistic distributions by [7] and on the Frèchet, Gumbel and Gompertz distributions by [13].

The second form is due to [8] although [14] also claimed to be the proponent. With a baseline *cdf* G(x), the CRT is given as:

$$F(x) = (1 + \alpha)G(x) + (\beta - \alpha)\, G^2(x) - \beta G^3(x) \qquad (2.2)$$

where $\alpha, \beta \in [-1, 1]$. This form was used on the Gompertz distribution by [15].

Another CRT is due to [9]. This form has the CRT of the form:

$$F(x) = (1 + \alpha + \beta)G(x) - (\alpha + 2\beta)\, G^2(x) + \beta G^3(x) \qquad (2.3)$$

where $\alpha \in [-1, 1]$ and $\beta \in [0, 1]$.

[10] also proposed another form of CRT with the *cdf* given as:

$$F(x) = (1-\alpha)G(x) + 3\alpha G^2(x) - 2\alpha G^3(x) \quad (2.4)$$

where $\alpha \in [-1, 1]$. It was used to generalize the uniform distribution by [10].

Another form was proposed by [11] with cdf given as:

$$F(x) = (1+\alpha)G(x) - 2\alpha G^2(x) + \alpha G^3(x) \quad (2.5)$$

where $\alpha \in [-1, 1]$

III. Cubic Rank Transmuted Exponential (CRTE) distribution

The *cdf* of a random variable X with the exponential distribution is given as:

$$G(x) = 1 - e^{-\lambda x} \quad (3.0)$$

A. The *cdf* of the CRTE distributions

Inserting (3.0) into (2.1), (2.2), (2.3), (2.4), and (2.5) respectively give the *cdf* of the CRTE distributions due to [7, 8, 9, 10, 11]. The *cdf* of these CRTEs are respectively given as:

$$F(x) = 1 + (\alpha+\beta-3)e^{-\lambda x} + (3-\alpha-2\beta)e^{-2\lambda x} + (\beta-1)e^{-3\lambda x} \quad (3.1.1)$$

$$F(x) = 1 + (\alpha+\beta-1)e^{-\lambda x} - (\alpha+2\beta)e^{-2\lambda x} + \beta e^{-3\lambda x} \quad (3.1.2)$$

$$F(x) = 1 - (1-\alpha)e^{-\lambda x} - (\alpha-\beta)e^{-2\lambda x} - \beta e^{-3\lambda x} \quad (3.1.3)$$

$$F(x) = 1 - (1-\alpha)e^{-\lambda x} - 3\alpha e^{-2\lambda x} + 2\alpha e^{-3\lambda x} \quad (3.1.4)$$

$$F(x) = 1 - e^{-\lambda x} + \alpha e^{-2\lambda x} - \alpha e^{-3\lambda x} \quad (3.1.5)$$

B. The *pdf* of the CTRE distributions

The probability distribution functions of the five CRTE distributions are obtained by differentiating each *cdf*. These *pdfs* are given as:

$$f(x) = \lambda e^{-\lambda x}\left(3-\alpha-\beta+(2\alpha+4\beta-6)e^{-\lambda x} - (3\beta-3)e^{-2\lambda x}\right) \quad (3.2.1)$$

$$f(x) = \lambda e^{-\lambda x}\left(1-\alpha-\beta+4\beta e^{-\lambda x} + 2\alpha e^{-\theta x} - 3\beta e^{-2\lambda x}\right) \quad (3.2.2)$$

$$f(x) = \lambda e^{-\lambda x}\left(1-\alpha+2(\alpha-\beta)e^{-\lambda x} + 3\beta e^{-2\lambda x}\right) \quad (3.2.3)$$

$$f(x) = \lambda e^{-\lambda x}\left(1-\alpha+6\alpha e^{-\lambda x} - 6\alpha e^{-2\lambda x}\right) \quad (3.2.4)$$

$$f(x) = \lambda e^{-\lambda x}\left(1-2\alpha e^{-\lambda x} + 3\alpha e^{-2\lambda x}\right) \quad (3.2.5)$$

C. Survival Functions

The survival function, *S(x)* of the five CRTE distributions are:

$$S(x) = 1 - \alpha\left(1-e^{-\lambda x}\right) - (\beta-\alpha)\left(1-e^{-\lambda x}\right)^2 - (1-\beta)\left(1-e^{-\lambda x}\right)^3 \quad (3.3.1)$$

$$S(x) = (1-\alpha-\beta)e^{-\lambda x} + (\alpha+2\beta)e^{-2\lambda x} - \beta e^{-3\lambda} \quad (3.3.2)$$

$$S(x) = (1-\alpha)e^{-\lambda x} + (\alpha-\beta)e^{-2\lambda x} + \beta e^{-3\lambda x} \quad (3.3.3)$$

$$S(x) = (1-\alpha)e^{-\lambda x} - 2\alpha e^{-3\lambda x} + 3\alpha e^{-2\lambda} \quad (3.3.4)$$

$$S(x) = e^{-\lambda x} - e^{-2\lambda x} + \alpha e^{-3\lambda x} \quad (3.3.5)$$

D. Hazard Functions

The hazard function, *h(x)* of the five CRTE distributions are:

$$h(x) = \frac{\alpha\lambda e^{-\lambda x} + 2\lambda(\beta-\alpha)\left(1-e^{-\lambda x}\right)e^{-\lambda x} + 3\lambda(1-\beta)\left(1-e^{-\lambda x}\right)^2 e^{-\lambda x}}{1-\alpha\left(1-e^{-\lambda x}\right) + (\beta-\alpha)\left(1-e^{-\lambda x}\right)^2 + (1-\beta)\left(1-e^{-\lambda x}\right)^3} \quad (3.4.1)$$

$$h(x) = \frac{\lambda\left(1-\alpha-\beta+4\beta e^{-\lambda x} + 2\alpha e^{-\lambda x} - 3\beta e^{-2\lambda x}\right)}{1-\alpha-\beta-\beta e^{-2\lambda x} + 2\beta e^{-\lambda x} + \alpha e^{-\lambda x}} \quad (3.4.2)$$

$$h(x) = \frac{\lambda\left(1-\alpha+3\beta e^{-2\lambda x} + 2\alpha e^{-\lambda x} - 2\beta e^{-\lambda x}\right)}{1-\alpha+\beta e^{-2\lambda} + \alpha e^{-\lambda x} - \beta e^{-\lambda x}} \quad (3.4.3)$$

$$h(x) = \frac{\lambda\left(\alpha-1-6\alpha e^{-\lambda x} + 6\alpha e^{-2\lambda x}\right)}{\alpha-1-3\alpha e^{-\lambda x} + 2\alpha e^{-2\lambda x}} \quad (3.4.4)$$

$$h(x) = \frac{\left(\lambda - 2\alpha e^{-\lambda x} + 3\alpha\lambda e^{-2\lambda x}\right)}{1-\alpha e^{-\lambda x} + e^{-2\lambda x}} \quad (3.4.5)$$

E. Order Statistics

Given that $X_{1,n} < X_{2,n} < \ldots < X_{n,n}$ is a set of ordered random variable of size n, the m^{th} order statistics of random variable X for the five CRTE distributions are respectively obtained as:

$$f_{m,n}(x) = \frac{n!}{(m-1)!(n-m)!}\left[\lambda e^{-\lambda x}\left(3-\alpha-\beta-(3\beta-3)e^{-2\lambda x} + (2\alpha+4\beta-6)e^{-\lambda x}\right)\right]\left[1+(\alpha+\beta-3)e^{-\lambda x} + (3-\alpha-2\beta)e^{-2\lambda x} + (\beta-1)e^{-3\lambda x}\right]^{m-1}\left[-(\alpha+\beta-3)e^{-\lambda x} - (3-\alpha-2\beta)e^{-2\lambda x} - (\beta-1)e^{-3\lambda x}\right]^{n-m} \quad (3.5.1)$$

$$f_{m,n}(x) = \frac{n!}{(m-1)!(n-m)!}\left[\lambda e^{-\lambda x}\left(1-\alpha-\beta+4\beta e^{-\lambda x} + 2\alpha e^{-\lambda x} - 3\beta e^{-2\lambda x}\right)\right]\left[1+\beta e^{-3\lambda x} - (\alpha+2\beta)e^{-2\lambda x} + (\alpha+\beta-1)e^{-\lambda x}\right]^{m-1}\left[(\alpha+2\beta)e^{-2\lambda x} - \beta e^{-3\lambda x} - \right.$$

$$(\alpha+\beta-1)e^{-\lambda x}\big]^{n-m} \qquad (3.5.2)$$

$$f_{m,n}(x)=\frac{n!}{(m-1)!(n-m)!}\left[\lambda e^{-\lambda}\left(1-\alpha+2\alpha e^{-\lambda}-2\beta e^{-\lambda x}+3be^{-2\lambda x}\right)\right]\left[1-e^{-\lambda x}+\alpha e^{-\lambda x}-\alpha e^{-2\lambda x}+\beta e^{-2\lambda}-\beta e^{-3\lambda x}\right]^{m-1}\left[e^{-\lambda}-\alpha e^{-\lambda x}+\alpha e^{-2\lambda x}-\beta e^{-2\lambda x}+\beta e^{-3\lambda x}\right]^{n-m} \qquad (3.5.3)$$

$$f_{m,n}(x)=\frac{n!}{(m-1)!(n-m)!}\left[\lambda e^{-\lambda x}\left(1-\alpha+6\alpha e^{-\lambda x}-6\alpha e^{-2\lambda x}\right)\right]\left[1-e^{-\lambda x}+\alpha e^{-\lambda x}-3\alpha e^{-2\lambda x}+2\alpha e^{-3\lambda}\right]^{m-1}\left[e^{-\lambda x}-\alpha e^{-\lambda x}+3\alpha e^{-2\lambda x}-2\alpha e^{-3\lambda x}\right]^{n-m} \qquad (3.5.4)$$

$$f_{m,n}(x)=\frac{n!}{(m-1)!(n-m)!}\left[\lambda e^{-\lambda x}\left(1-2\alpha e^{-\lambda x}+3\alpha e^{-2\lambda x}\right)\right]\left[1-e^{-\lambda x}+\alpha e^{-2\lambda x}-\alpha e^{-3\lambda x}\right]^{m-1}\left[e^{-\lambda x}-\alpha e^{-2\lambda x}+\alpha e^{-3\lambda x}\right]^{n-m} \qquad (3.5.5)$$

F. Moment Generating Functions

The moment generating function for the five CRTE distributions are respectively obtained as:

$$E(e^{tx})=\frac{3\lambda}{\lambda-t}-\frac{\alpha\lambda}{\lambda-t}-\frac{\beta\lambda}{\lambda-t}-\frac{(3\beta-3)\lambda}{3\lambda-t}+\frac{(2\alpha+4\beta-6)\lambda}{2\lambda-t} \qquad (3.6.1)$$

$$E(e^{tx})=\frac{\lambda}{\lambda-t}-\frac{\alpha\lambda}{\lambda-t}-\frac{\beta\lambda}{\lambda-t}+\frac{4\beta\lambda}{2\lambda-t}+\frac{2\alpha\lambda}{2\lambda-t}-\frac{3\beta\lambda}{3\lambda-t} \qquad (3.6.2)$$

$$E(e^{tx})=\frac{\lambda}{\lambda-t}-\frac{\alpha\lambda}{\lambda-t}+\frac{2\alpha\lambda}{2\lambda-t}-\frac{2\beta\lambda}{2\lambda-t}+\frac{3\beta\lambda}{3\lambda-t} \qquad (3.6.3)$$

$$E(e^{tx})=\frac{\lambda}{\lambda-t}-\frac{\alpha\lambda}{\lambda-t}+\frac{6\alpha\lambda}{2\lambda-t}-\frac{6\alpha\lambda}{3\lambda-t} \qquad (3.6.4)$$

$$E(e^{tx})=\frac{\lambda}{\lambda-t}-\frac{2\alpha\lambda}{2\lambda-t}+\frac{3\alpha\lambda}{3\lambda-t} \qquad (3.6.5)$$

G. Raw Moments

$$E(x^k)=\left(3-\alpha-\beta-\frac{(\beta-1)}{3^k}+\frac{(\alpha+2\beta-3)}{2^k}\right)\frac{k!}{\lambda^k} \qquad (3.7.1)$$

$$E(x^k)=\left(1-\alpha-\beta+\frac{2\beta}{2^k}+\frac{\alpha}{2^k}-\frac{\beta}{3^k}\right)\frac{k!}{\lambda^k} \qquad (3.7.2)$$

$$E(x^k)=\left(1-\alpha+\frac{\alpha}{2^k}-\frac{\beta}{2^k}+\frac{\beta}{3^k}\right)\frac{k!}{\lambda^k} \qquad (3.7.3)$$

$$E(x^k)=\left(1-\alpha+\frac{3\alpha}{2^k}-\frac{2\alpha}{3^k}\right)\frac{k!}{\lambda^k} \qquad (3.7.4)$$

$$E(x^k)=\left(1-\frac{\alpha}{2^k}+\frac{\alpha}{3^k}\right)\frac{k!}{\lambda^k} \qquad (3.7.5)$$

H. Likelihood Functions

$$n\log(\lambda)-\lambda\sum_{i=1}^{n}(x_i)+\sum_{i=1}^{n}\log\left(3-\alpha-\beta-(3\beta-3)e^{-2\lambda x}+(2\alpha+4\beta-6)e^{-\lambda x}\right) \qquad (3.8.1)$$

$$n\log(\lambda)-\lambda\sum_{i=1}^{n}(x_i)+\sum_{i=1}^{n}\log\left(1-\alpha-\beta+4\beta e^{-\lambda x}+2\alpha e^{-\lambda x}-3\beta e^{-2\lambda x}\right) \qquad (3.8.2)$$

$$n\log(\lambda)-\lambda\sum_{i=1}^{n}(x_i)+\sum_{i=1}^{n}\log\left(1-\alpha+2\alpha e^{-\lambda x}-2\beta e^{-\lambda x}+3\beta e^{-2\lambda}\right) \qquad (3.8.3)$$

$$n\log(\lambda)-\lambda\sum_{i=1}^{n}(x_i)+\sum_{i=1}^{n}\log\left(1-\alpha+6\alpha e^{-\lambda x}-6\alpha e^{-2\lambda x}\right) \qquad (3.8.4)$$

$$n\log(\lambda)-\lambda\sum_{i=1}^{n}(x_i)+\sum_{i=1}^{n}\log\left(1-2\alpha e^{-\lambda x}+3\alpha e^{-2\lambda x}\right) \qquad (3.8.5)$$

IV. SHAPES OF DIFFERENT CRTE DISTRIBUTIONS

Figures 1a-e show plots for the *pdf* of five CRTE distributions examined in this study for different parameters. The *pdf* plots can be decreasing, decreasing-increasing-decreasing, upside down bathtub, and negatively skewed. The CRTE due to [7] appears to show more flexibilities. Figures 2a-f show that the hazard rate for the five CRTE distributions can be constant, monotonic increasing, monotonic decreasing, bathtub, and upside down bathtub have shapes

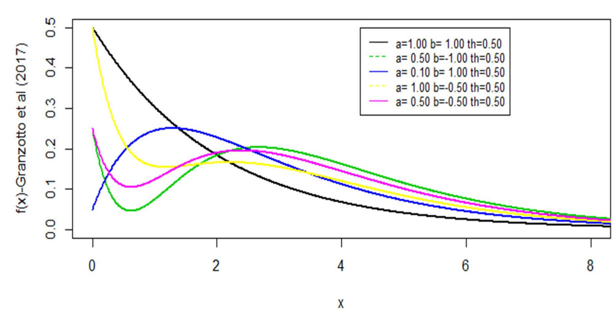

Figure 1a: pdf of Granzotto et al.(2017) CRTE distribution

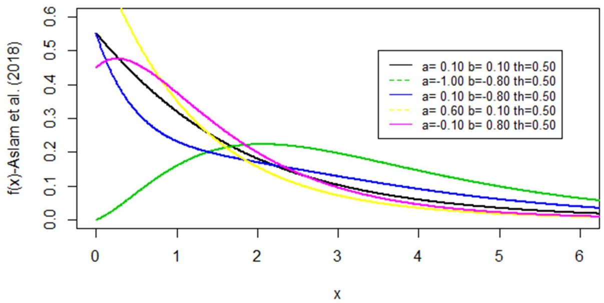

Figure 1b: pdf of Aslam et al. (2018) CRTE distribution

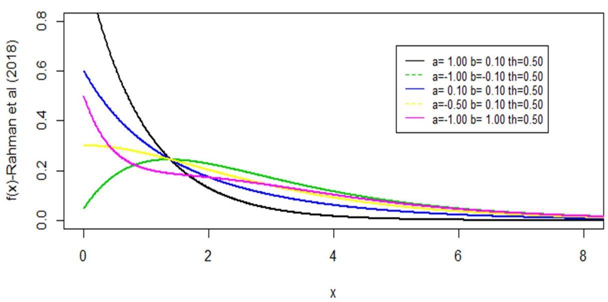

Figure 1c: pdf of Rahman et al (2018a) CRTE distribution

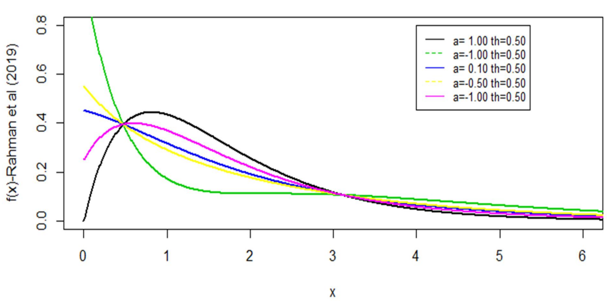

Figure 1d: pdf of Rahman et al (2019) CRTE distribution

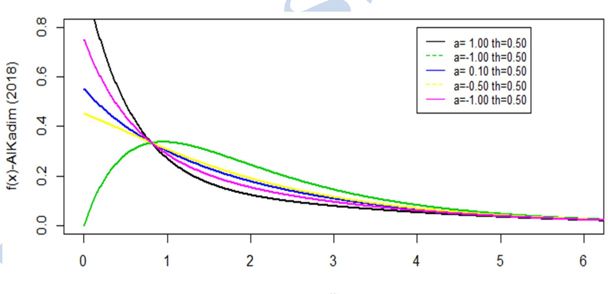

Figure 1e: pdf of Alkadim (2018) CRTE distribution

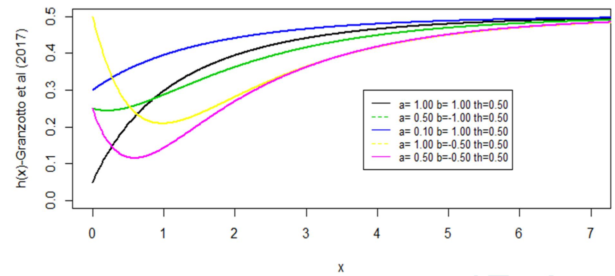

Figure 2a: hrf of Granzotto et al. (2017) CRTE distribution

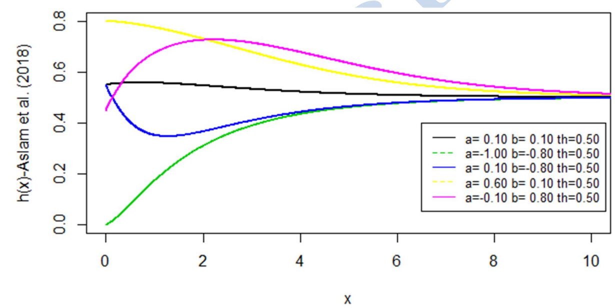

Figure 2b: hrf of Aslam et al. (2018) CRTE distribution

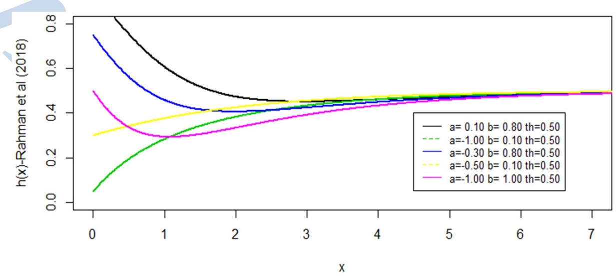

Figure 2c: hrf of Rahman et al (2018a) CRTE distribution

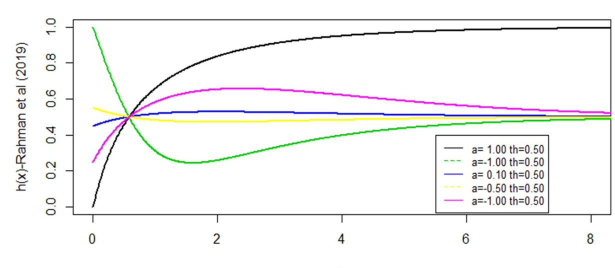

Figure 2d: hrf of Rahman et al (2019) CRTE distribution

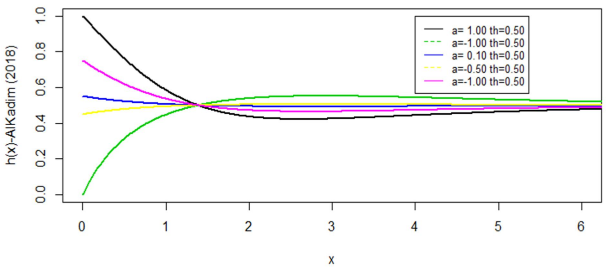

Figure 2e: hrf of Alkadim (2018) CRTE distribution

V. Application

This section applies the five CRTE distributions to the widely used remission times (month) of 128 bladder cancer patients as previously used by [16]. The result is presented in table 1. In selecting a good model, the study used the AIC (Akaike Information Criterion), CAIC (Corrected Akaike Information Criterion), and BIC (Bayesian information criterion). Distribution with the lowest value is adjudged the best.

0.08, 2.09, 3.48,4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57,5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54,3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64,3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05,2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26,2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02,4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25,8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07,3.36, 6.93, 8.65, 12.63, 22.69

Table 1: Parameter estimates and model selection criteria for bladder cancer remission (in months)

CRTE	λ	α	β	AIC	CAIC	BIC
Granzotto *et al.*(2017)	0.100	0.609	1.873	829.314	829.410	837.870
Aslam *et al.* (2018)	0.100	-0.391	0.873	829.314	829.410	837.870
Rahman *et al* (2018a)	0.0996	0.482	-0.873	829.314	829.410	837.870
Rahman *et al* (2019)	0.1021	0.436		827.337	827.369	833.042
Alkadim (2018)	0.1186	-0.549		828.858	828.890	834.563

VI. Conclusion

With the lowest AIC, CAIC, and BIC, the CRTE distribution using [10] transmutation map provides the best fit for the data. This is followed by the CRTE due to [11] transmutation map. It can also be observed that the estimated parameter (λ) for CRTE due to [7] and [8] are the same while the transmutation parameter $\boldsymbol{\beta}$ = - $\boldsymbol{\beta}$ for the CRTE using [8] and [9].

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