Construction of Connected Incomplete Block Designs for Statistical Experimental Layouts

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***Abstract* - In this study, seventeen (17) Incomplete Block Designs (IBDs) namely; designs G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V and W constructed from the initial blocks due to Nguyen, (1994) via the cyclic method and JAVA codes are presented as an extension of the existing designs A-F. All the 17 IBDs were investigated for connectedness and thereafter categorized. The numbers of pairwise treatment connections in blocks were obtained and their concurrence graphs plotted for all the constructed IBDs. The study revealed that the newly constructed design Q, turned out to be better than Nguyen's designs A-F with 28 shorter paths and is closely followed by designs K, N, O, P, S, V and W with 27 shorter-paths each. Incidentally, Nguyen's design A turns out to be the least connected design among all the IBDs under consideration. The study concluded that the overall best design is design Q based on connectedness and hence recommends- designs Q for use with their experimental layouts by experimenters, investigators or evaluators who seek for the best optimal design of size t = 9, b = 9, k = 3, r = 3 in any field experiment based on connectedness. This experimental layout is equally recommended in the treatment plan of identified cases of Severe Acute Malnutrition (SAM), where nine feeding volumes (9) with three (3) replicate in a block size of three (3) in a Ready-to-use-therapeutic-feeding program is desirable**

Keywords: *Incomplete block design; pairwise treatment; concurrence graph; connectedness; optimal design; incidence matrix.*

I. Introduction

It is important that any well-designed experiment should be able to give answers to scientific questions clearly, concisely and efficiently at the initial instance. The analysis may require just more than a few basic plots with evidence of responses as a result of transformation, depending on the available factors under consideration. Therefore, for any well-designed experiment to be good enough, it is expected that the variance and the covariance of the expected parameters of interest will turn out to be small Atkinson (2014). A properly designed experiment for a particular research objective is the basis of all successful experiments Sewenet (2019).

In many experimental situations, the number of treatments to be applied to the available blocks may be too large, making it necessary for a large number of blocks to take all the treatments at a time together. This may certainly increase the cost of experimentation in terms of funding to cover cost of labor, time; etc. The completely randomized design (CRD) and randomized block design (RBD) may not be ideal in such situations because they will require a large number of experimental units to take all the treatments. In situations where it is clear that the numbers of homogeneous experimental units are sufficient to accommodate all the treatments in a block, then the incomplete block designs (IBD) become the preferred alternative. This is because of its flexibility in allowing each block to receive only some and not all the treatments to be compared Toutenburg and Shalabh (2009).

An incomplete-block design refers to a block design in which the size of each block is smaller than the number of treatments Chigbu (1998). Usually, the total number of plots in an incomplete-block design having r_i replications while administering t_i treatment in b_j block and of size k_j

such that:

$$\sum_i r_{i.} = \sum_j k_j \quad (1)$$

is structurally represented:

$$y_{ijk} = a_0 + t_i + b_j + e_{ijk}, \quad k = 1,2,\ldots n_{ij} \quad (2)$$

where y_{ijk} response in the a_0 constant, t_i and b_j are the i^{th} treatment and the j^{th} block effects, respectively, e_{ijk} are the random error, r_i and k_j are the number of replications of t_i and block b_j respectively, n_{ij} is the number of plots in the j^{th} block that receives the i^{th} treatment Onukogu and Chigbu (2002).

In most experiments, especially the designed experiments in Agriculture, Industries, Pharmaceuticals, Medicine, Nutrition, Hydrology etc. the efficiency of Fisher's very popular Randomized Block Designs (RBD's) and Latin Square Designs (LSD) had been found to lose its efficiency when the number of plots per block or row and column increases to say, ten and above Bose and Nair (1939).

An incomplete block design of size (*v, k, r*), where *v* is treatment and set out in *b*- blocks of size *k(< v)* where the layout is in such a way that the each treatment is replicated *r-times* Cheng and Bailey (1991). The assumption here is that no treatment occurs more than once in a block. The information matrix for the adjusted treatment effects of the IBD of size (*v, k, r*) is as given below:

$$\underline{L} = rI - \left(\frac{1}{K}\right)NN' \quad (3)$$

where NN' is the treatment concurrence matrix with a correspondence treatment concurrence graph (Nguyen, 1994).

II. Methodology

1. Combinatorial Justification for connectedness

According to Eccleston and Hedayat (1974) connectedness is an important property which every block design must possess if it is to provide an unbiased estimator for all elementary treatment contrasts under the usual linear additive model. They went further to classify the family of connected designs into three sub-classes namely: locally connected, globally connected and pseudo-globally connected designs. They opined that, a locally connected design is one in which not all the observations participate in the estimation. A globally connected design is one in which all observations participate in the estimation. Finally, a pseudo-globally connected design is a compromise between locally and globally connected designs. Theorems and corollaries were given which characterize the different classes of connected designs. In their discussion on the optimality of connected designs, they showed that there is much to be gained by partitioning the family of connected designs in the above fashion. They identified an optimality criteria such as the S-optimality which was suggested by Shah, which selects the design with minimum trace of the information matrix squared and (M, S)-optimality which selects the S-optimal design from the class of designs with maximum trace of the information matrix. Using these optimality criteria, they were able to derive some new results which they hope to be of interest to the users and researchers in the field of optimum design theory.

Cameron (2006) in the Encyclopedia of design theory asserted that a block design is said to be connected if the rank of C is *v-1* (that is, the null space of C is spanned by *j*), if this holds, then the image of C consist of all the vectors in $\Re^{+}$ with coordinate sum 0. Similarly, Chigbu and Ekpo (2009) gave new findings regarding the circuits and automorphism grouping of three (4x4)/4 semi-Latin squares which are equally A-, D- and E- optimal based on designs whose concurrence graphs had fewer number of shortest circuits.

2. Basis for the Construction of the IBD

The constructing the IBD is based on the concept of symmetrically repeated differences. By the use of this concept, it is possible to construct the entire design with the help of the initial blocks. The discovery of the initial blocks is very much facilitated by the use of the properties of the primitive roots of the binomial equations in the Galois field, but the use of these properties is not essentially here Mahanta (2018). Whenever, the initial blocks can be obtained by trial or otherwise, satisfying the condition that the differences are symmetrically repeated, a design construction is obtainable Bose (2011).

The construction of a cyclic balanced incomplete block design requires the use of the initial block. However, the choice of the initial block is quite arbitrary, in that, it would lead to the construction of an appropriate design by a method of cyclical substitution. For convenience, the initial block will be taken to be the block with the lowest numerical values. Consider an IBD of size

$(t=7, b=7, r=3, k=3, \lambda=1)$ for example, and using block (0, 1, 3) as the initial block will give the following seven incomplete blocks:
(0 1 3), (1 2 4),(2 3 5),(3 4 6), (4 5 0),(5 6 1),(6 0 2).

Here, treatment is replicated 3 times and every pair of treatments occurs together just ones in a block. The cyclic method of construction is such that a block is obtained just by adding one to each element in the previous block and reducing modulo 7 when necessary. Suppose an initial block $\{i_1, i_2, \ldots, i_k\}$ is given after labeling the treatment by the modulo t. The other blocks of the cyclic design are $\{i_1+1, i_2+1, \ldots, i_k+1\}$, and so on, with all arithmetic done in modulo t. This cyclic construction method works if $b = t$. Every choice of an initial block has a difference table as the one constructed below John (1987) ; Bailey and Cameron (2019):

$$\begin{array}{c|cccc} & i_1 & i_2 & \cdots & i_k \\ \hline i_1 & 0 & i_2 - i_1 & \cdots & i_k - i_1 \\ i_2 & i_1 - i_2 & 0 & \cdots & i_k - i_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ i_k & i_1 - i_k & i_2 - i_k & \cdots & 0 \end{array} \qquad (4)$$

The cyclic method of construction of designs is such that a block is obtained by adding one to each element in the previous block and reducing to appropriate modulo when necessary.

3. Method of Construction of the IBDs

In dealing with all of the construction in this paper, the JAVA Codes and MATLAB software were extensively utilized, especially in the operations involving all matrices obtained from the incidence and the concurrence while the concurrence graphs were drawn with the aid of Coral draw.

Existing Set of Six Incomplete Block Designs Due to Nguyen, 1994

This paper discovered and utilized a set of six IBDs of size (t = b = 9, r = 3, k = 3) constructed by Nguyen in 1994 with treatments greater than 14 in all the examples that motivated his construction. The constructed designs could be used for experimentation that involved the removal of lichens from asbestos cement (AC), for testing the weathering stability of some brands of paints under ultraviolet (UV) irradiation; where the UV weather meter could only hold three paints panel at a time. However, in illustrating the steps of his algorithm for the construction of the IBD of size (t = b = 9, r = 3, k = 3), he did not involve the use of combinatorial properties such as the design connectedness/circuits via concurrence graphs of the design. The six existing constructed incomplete block designs (IBDs) of size (t = b = 9, r = 3, k = 3) due to Nguyen (1994); namely: designs *A, B, C, D, E* and *F*, having seventeen initial blocks with each design containing *9* treatments in *9* blocks, *3* plots; and with *3* replications are as given below in Table 1.

Table 1: Existing Incomplete Block Designs Due to Nguyen (1994)

3	1	7
9	8	4
5	2	6
3	2	7
5	9	8
6	1	4
5	9	4
2	8	3
7	1	6

(*A*)

3	1	9
7	8	4
5	2	6
3	2	7
5	9	8
6	1	4
5	9	4
2	8	3
7	1	6

(*B*)

3	1	9
7	8	4
5	2	6
9	2	7
5	3	8
6	1	4
5	9	4
2	8	3
7	1	6

(*C*)

3	1	9
7	8	4
5	2	6
3	2	7
5	3	8
6	1	4
5	9	4
2	8	9
7	1	6

(*D*)

3	1	9
7	8	4
5	2	6
1	2	7
5	3	8
6	1	4
5	9	4
2	8	9
7	3	6

(*E*)

3	1	9
7	8	4
5	2	6
9	2	7
5	3	8
6	1	4
5	9	4
2	8	1
7	3	6

(*F*)

Note: (A) - (F) signifies name of designs, e.g., (A) is Design A

III. Applications

Newly Constructed Set of Incomplete Block Designs Using all the initial blocks in Nguyen, (1994).

The cyclic method of constructing incomplete block designs was used here in the construction of the seventeen new IBDs of the same size *(9, 3, 3)* as those of Nguyen (1994), i.e., designs *G, H, I, J, K ,L, M, N, O, P, Q, R, S, T, U, V and W*. Design *G* was constructed using Nguyen's initial block *(3 1 7)* while design *H* was also constructed using Nguyen's initial block *(3 1 9),* design *I* with initial block *(9 8 4),* design *J* with initial block *(7 8 4),* design *K* with initial block *(9 8 4),* design *L* with initial block *(3 2 7),* design *M* with initial block *(9 2 7),* design *N* with initial block *(1 2 7),* design *O* with initial block *(5 9 8),* design *P* with initial block *(5 3 8),* design *Q* with initial block *(6 1 4),* design *R* with initial block *(5 9 4),* design *S* with initial block *(2 8 3),* design *T* with initial block *(2 8 9),* design *U* with initial block *(2 8 1),* design *V* with initial block *(7 1 6)* and design *W* with initial block *(7 3 6).* The newly constructed incomplete block designs are as given in Table 2.

Table 2: Newly Constructed Set of Seventeen Incomplete Block Designs Using all Nguyen (1994) blocks as initial Blocks

3	1	7	3	1	9	9	8	4	7	8	4	5	2	6	3	2	7
4	2	8	4	2	1	1	9	5	8	9	5	6	3	7	4	3	8
5	3	9	5	3	2	2	1	6	9	1	6	7	4	8	5	4	9
6	4	1	6	4	3	3	2	7	1	2	7	8	5	9	6	5	1
7	5	2	7	5	4	4	3	8	2	3	8	9	6	1	7	6	2
8	6	3	8	6	5	5	4	9	3	4	9	1	7	2	8	7	3
9	7	4	9	7	6	6	5	1	4	5	1	2	8	3	9	8	4
1	8	5	1	8	7	7	6	2	5	6	2	3	9	4	1	9	5
2	9	6	2	9	8	8	7	3	6	7	3	4	1	5	2	1	6
	(G)			(H)			(I)			(J)			(K)			(L)	

9	2	7	1	2	7	5	9	8	5	3	8	6	1	4	5	9	4
1	3	8	2	3	8	6	1	9	6	4	9	7	2	5	6	1	5
2	4	9	3	4	9	7	2	1	7	5	1	8	3	6	7	2	6
3	5	1	4	5	1	8	3	2	8	6	2	9	4	7	8	3	7
4	6	2	5	6	2	9	4	3	9	7	3	1	5	8	9	4	8
5	7	3	6	7	3	1	5	4	1	8	4	2	6	9	1	5	9
6	8	4	7	8	4	2	6	5	2	9	5	3	7	1	2	6	1
7	9	5	8	9	5	3	7	6	3	1	6	4	8	2	3	7	2
8	1	6	9	1	6	4	8	7	4	2	7	5	9	3	4	8	3
	(M)			(N)			(O)			(P)			(Q)			(R)	

2	8	3	2	8	9	2	8	1	7	1	6	7	3	6
3	9	4	3	9	1	3	9	2	8	2	7	8	4	7
4	1	5	4	1	2	4	1	3	9	3	8	9	5	8
5	2	6	5	2	3	5	2	4	1	4	9	1	6	9
6	3	7	6	3	4	6	3	5	2	5	1	2	7	1
7	4	8	7	4	5	7	4	6	3	6	2	3	8	2
8	5	9	8	5	6	8	5	7	4	7	3	4	9	3
9	6	1	9	6	7	9	6	8	5	8	4	5	1	4
1	7	2	1	7	8	1	7	9	6	9	5	6	2	5
	(S)			(T)			(U)			(V)			(W)	

Note: (G) - (W) signifies name of the designs, e.g., (G) is Design G, (H) is Design H etc.

IV RESULTS

4.1 Calculation of the Incidence Matrices

4.1.1 Results for Design (G) Due to Nguyen's Initial Block (3 1 7)

1	0	0	1	0	0	0	1	0
0	1	0	0	1	0	0	0	1
1	0	1	0	0	1	0	0	0
0	1	0	1	0	0	1	0	0
0	0	1	0	1	0	0	1	0
0	0	0	1	0	1	0	0	1
1	0	0	0	1	0	1	0	0
0	1	0	0	0	1	0	1	0
0	0	1	0	0	0	1	0	1

Figure 1: Incidence Matrix for Design G

4.1.2 Results for Design (H) Due to Nguyen's Initial Block (3 1 9)

1	1	0	0	0	0	0	1	0
0	1	1	0	0	0	0	0	1
1	0	1	1	0	0	0	0	0
0	1	0	1	1	0	0	0	0
0	0	1	0	1	1	0	0	0
0	0	0	1	0	1	1	0	0
0	0	0	0	1	0	1	1	0
0	0	0	0	0	1	0	1	1
1	0	0	0	0	0	1	0	1

Figure 2: Incidence Matrix for Design H

4.1.3 Results for Design (I) Due to Nguyen's Initial Block (9 8 4)

0	1	1	0	0	0	1	0	0
0	0	1	1	0	0	0	1	0
0	0	0	1	1	0	0	0	1
1	0	0	0	1	1	0	0	0
0	1	0	0	0	1	1	0	0
0	0	1	0	0	0	1	1	0
0	0	0	1	0	0	0	1	1
1	0	0	0	1	0	0	0	1
1	1	0	0	0	1	0	0	0

Figure 3: Incidence Matrix for Design I

4.1.4 Results for Design (J) Due to Nguyen's Initial Block (7 8 4)

$$
\begin{bmatrix}
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
$$

Figure 4: Incidence Matrix for Design J

4.1.5 Results for Design (K) Due to Nguyen's Initial Block (5 2 6)

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0
\end{bmatrix}
$$

Figure 5: Incidence Matrix for Design K

4.1.6 Results for Design (L) Due to Nguyen's Initial Block (3 2 7)

$$
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}
$$

Figure 6: Incidence Matrix for Design L

4.1.7 Results for Design (M) Due to Nguyen's Initial Block (9 2 7)

$$
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
$$

Figure 7: Incidence Matrix for Design M

4.1.8 Results for Design (N) Due to Nguyen's Initial Block (1 2 7)

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Figure 8: Incidence Matrix for Design N

4.1.9 Results for Design (O) Due to Nguyen's Initial Block (5 9 8)

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure 9: Incidence Matrix for Design O

4.1.10 Results for Design (P) Due to Nguyen's Initial Block (5 3 8)

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Figure 10: Incidence Matrix for Design P

4.1.11 Results for Design (Q) Due to Nguyen's Initial Block (6 1 4)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Figure 11: Incidence Matrix for Design Q

4.1.12 Results for Design (R) Due to Nguyen's Initial Block (5 9 4)

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Figure 12: Incidence Matrix for Design R

4.1.13 Results for Design (S) Due to Nguyen's Initial Block (2 8 3)

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Figure 13: Incidence Matrix for Design S

4.1.14 Results for Design (T) Due to Nguyen's Initial Block (2 8 9)

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Figure 14: Incidence Matrix for Design T

4.1.15 Results for Design (U) Due to Nguyen's Initial Block (2 8 1)

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Figure 15: Incidence Matrix for Design U

4.1.16 Results for Design (V) Due to Nguyen's Initial Block (7 1 6)

1	0	0	1	1	0	0	0	0
0	1	0	0	1	1	0	0	0
0	0	1	0	0	1	1	0	0
0	0	0	1	0	0	1	1	0
0	0	0	0	1	0	0	1	1
1	0	0	0	0	1	0	0	1
1	1	0	0	0	0	1	0	0
0	1	1	0	0	0	0	1	0
0	0	1	1	0	0	0	0	1

Figure 16: Incidence Matrix for Design V

4.1.17 Results for Design (W) Due to Nguyen's Initial Block (7 3 6)

0	0	0	1	1	0	0	1	0
0	0	0	0	1	1	0	0	1
1	0	0	0	0	1	1	0	0
0	1	0	0	0	0	1	1	0
0	0	1	0	0	0	0	1	1
1	0	0	1	0	0	0	0	1
1	1	0	0	1	0	0	0	0
0	1	1	0	0	1	0	0	0
0	0	1	1	0	0	1	0	0

Figure 17: Incidence Matrix for Design W

Plotting of concurrence graphs for the 17 constructed IBDs

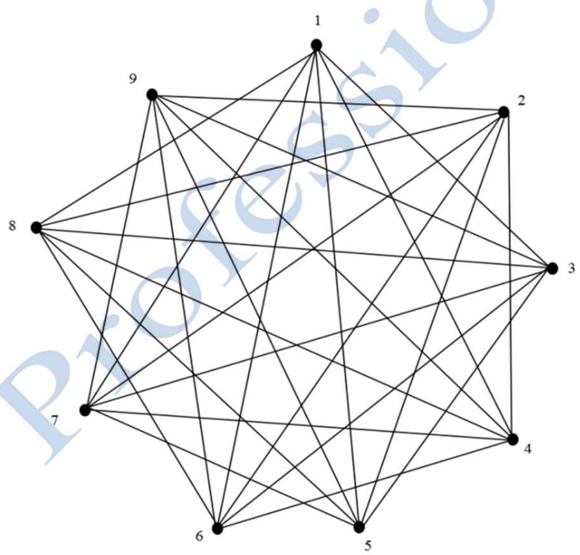

Figure 18: Concurrence Graph of Design G

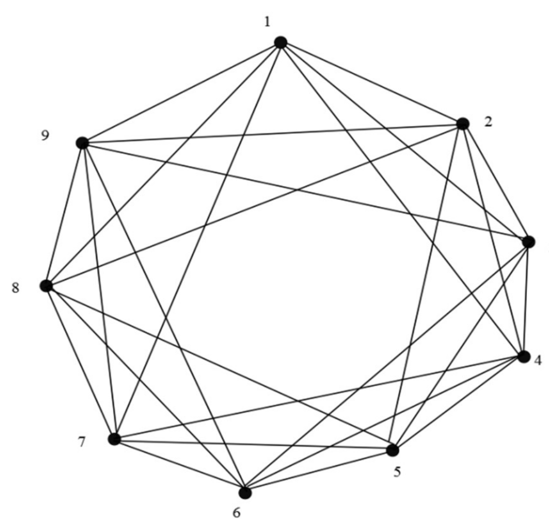

Figure 19: Concurrence Graph of Design H

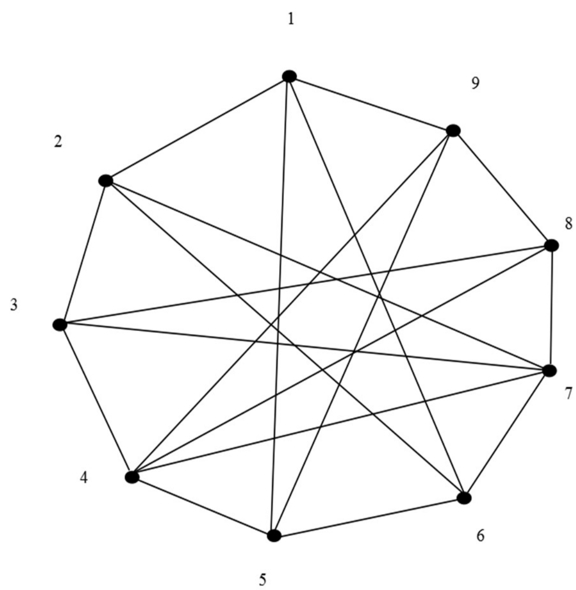

Figure 20: Concurrence Graph of Design I

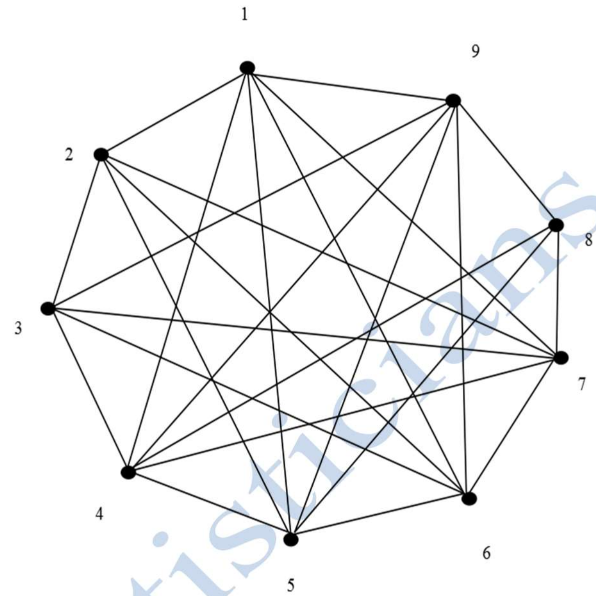

Figure 21: Concurrence Graph of Design J

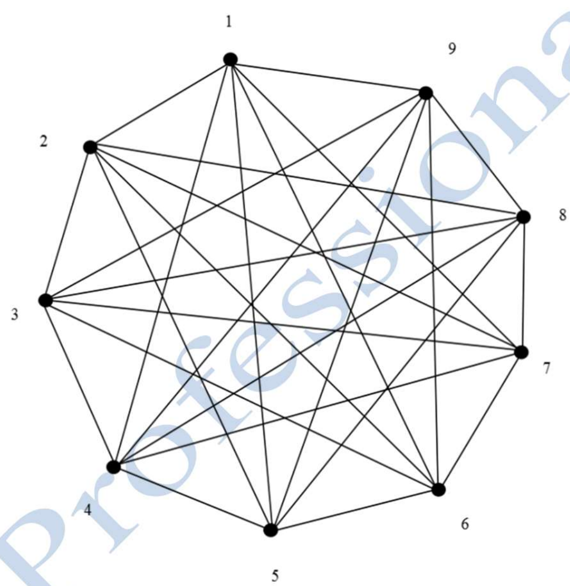

Figure 22: Concurrence Graph of Design K

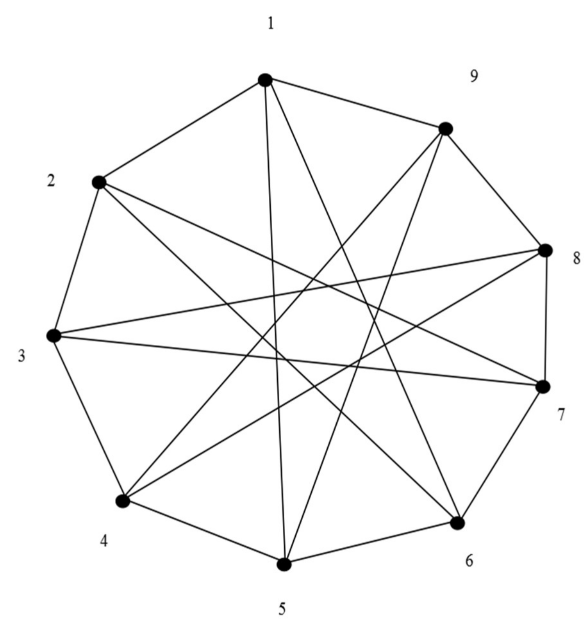

Figure 23: Concurrence Graph of Design L

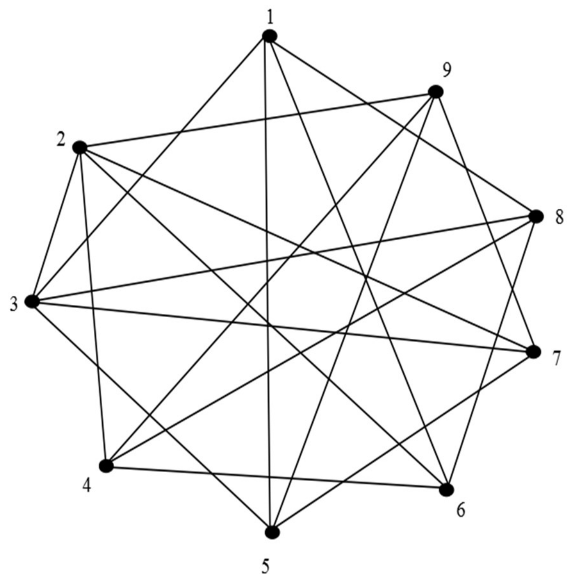

Figure 24: Concurrence Graph of Design M

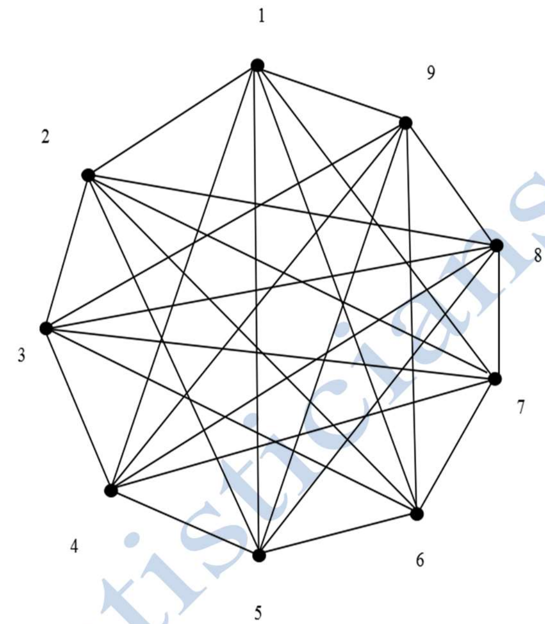

Figure 25: Concurrence Graph of Design N

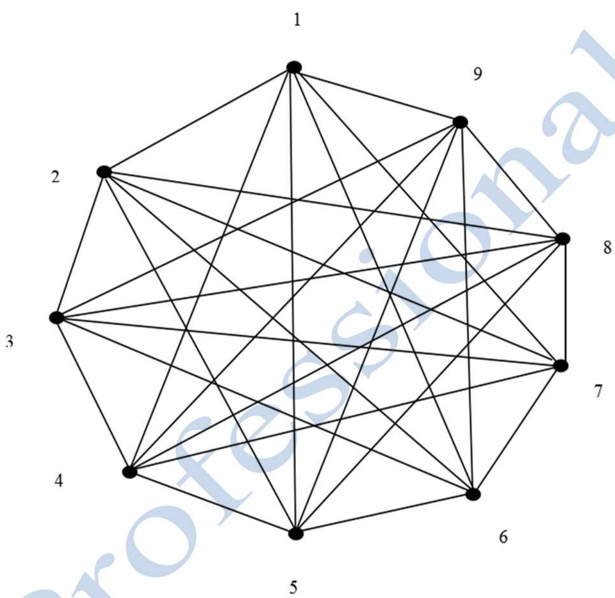

Figure 26: Concurrence Graph of Design O

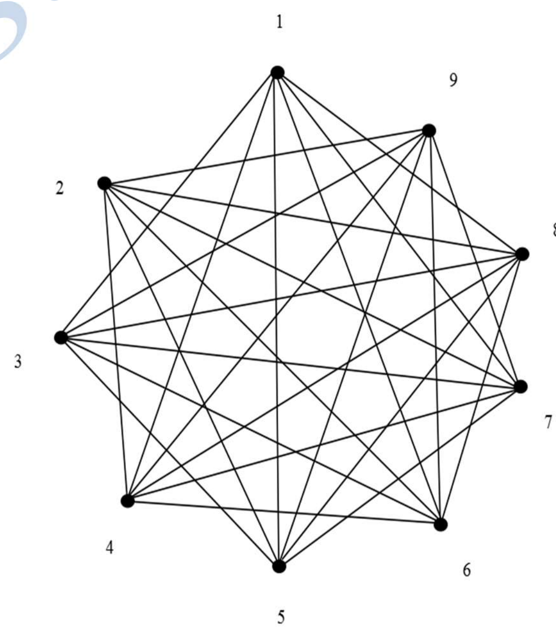

Figure 27: Concurrence Graph of Design P

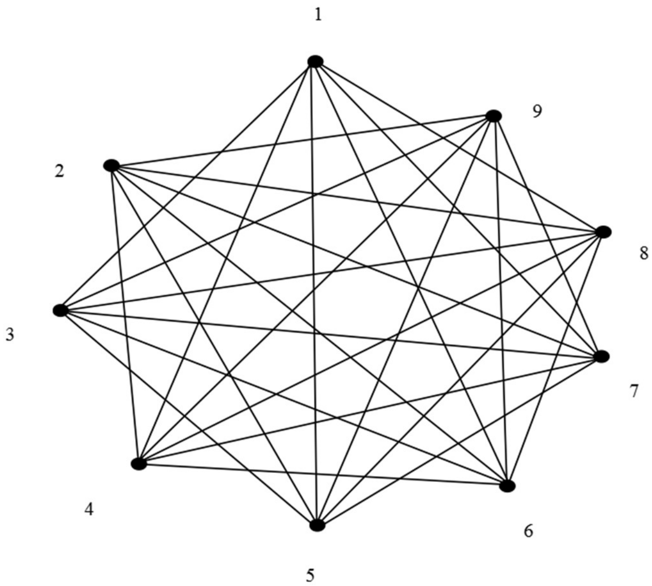

Figure 28: Concurrence Graph of Design Q

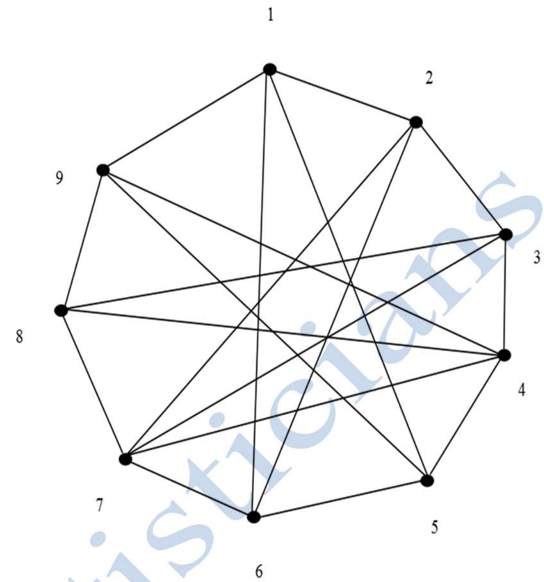

Figure 29: Concurrence Graph of Design R

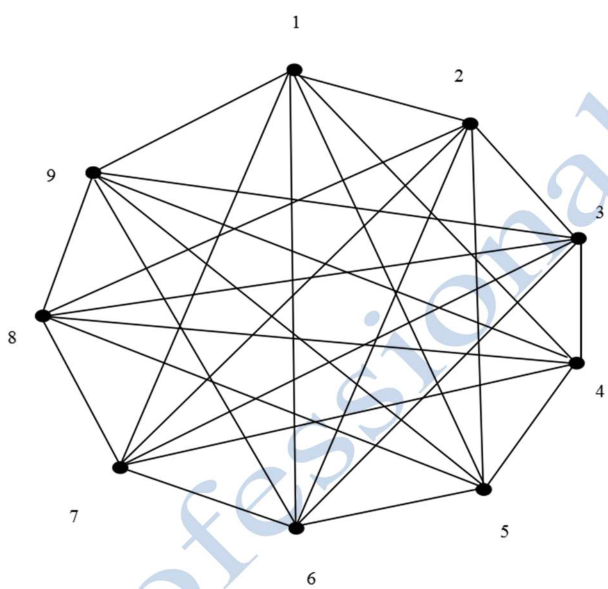

Figure 30: Concurrence Graph of Design S

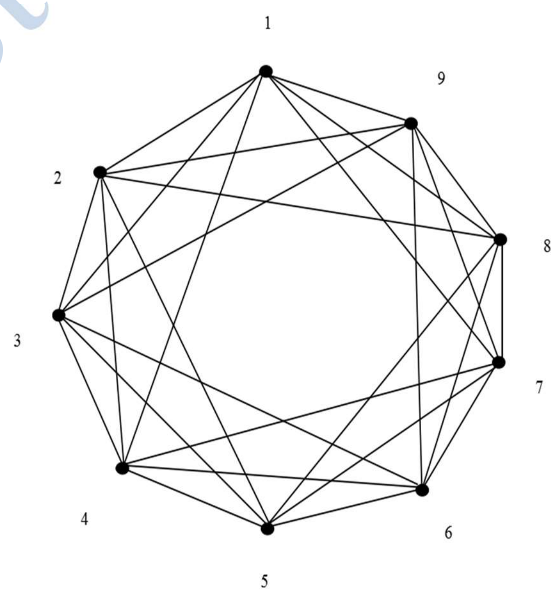

Figure 31: Concurrence Graph of Design T

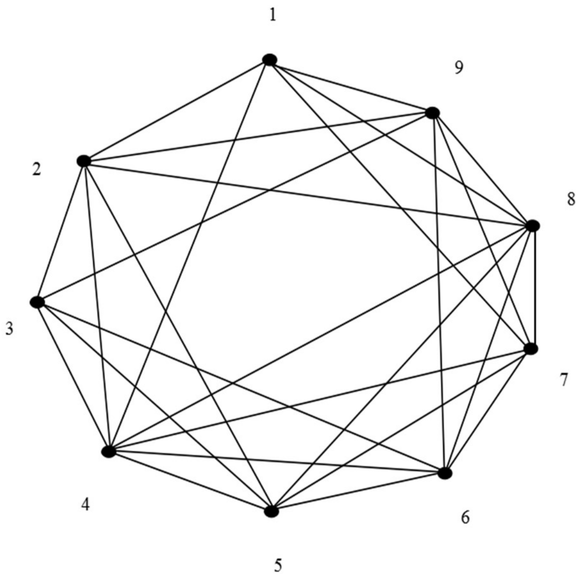

Figure 32: Concurrence Graph of Design U

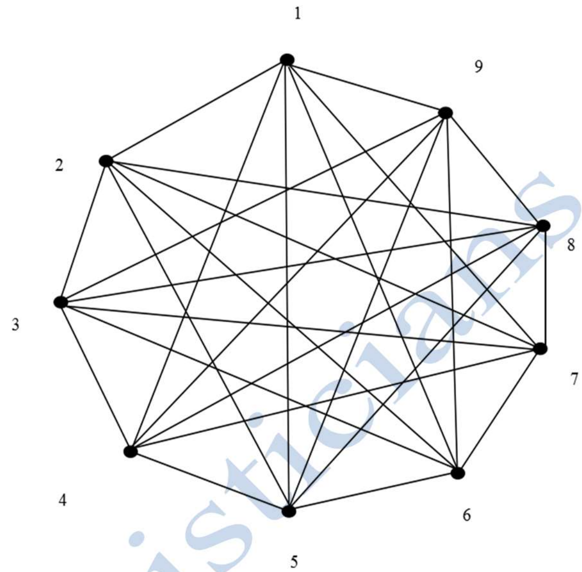

Figure 33: Concurrence Graph of Design V

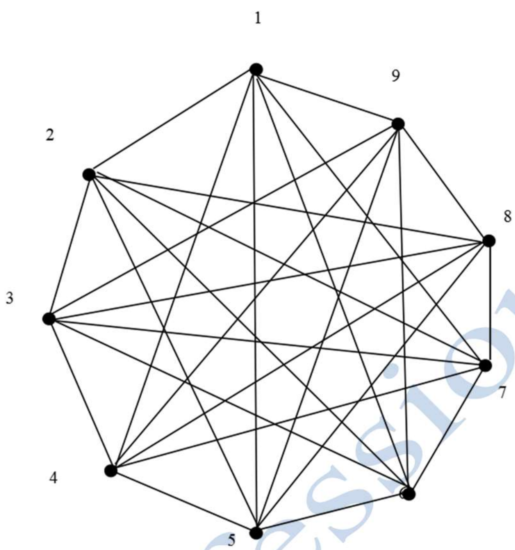

Figure 34: Concurrence Graph of Design W

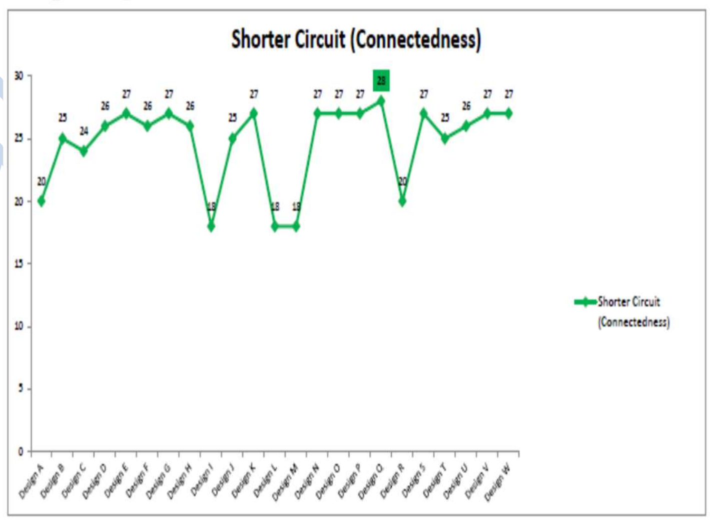

Figure 35: Graph of Design A-W showing design path-lengths

V. CONCLUSION

In terms of shorter-paths which implies minimum variance as shown in figures 18-34 design Q, one of the newly constructed designs turned out to be better than the others with 28 shorter paths (minimum variance of 1 path) as shown in figure 35, this is closely followed by designs K, N, O, P, S, V and W with 27 shorter-paths. Design Q has the least number (8) of longer paths with (0) longest path; hence the best design for experimentation in terms of connectedness. This result was collaborated by Bailey (2009) where it was stated that for the difference between the effects of treatments *i* and *j*, one should ensure that the variance of the estimator of this difference is small enough. Thus the primary interest is therefore, to explore when properties of the variances can be deduced from an examination of the graph itself without necessarily calculating the generalized inverse of a matrix; and that the variance decreases as concurrence increases or that variance increases with distance as shown in the concurrence graphs.

Again, this finding was collaborated by the findings of Eccleston and Hedayat (1974) which asserted that connectedness is an important property which every block design must possess if it is to provide an unbiased estimator for all elementary treatment contrasts under the usual linear additive model. Similarly, finding by Lindsay (1983) further supports the result obtained here, with the assertion which described the connection between the efficiency of incomplete block designs and numbers of paths of different lengths in their variety concurrence graphs offer in their own right, an intuitively cogent way of assessing designs; to show that they provide reasonable approximations to the conventional harmonic mean efficiency factor with emphasis on statistical motivation and on the capacity of graph theory to simplify; while algebraic details were kept to a minimum.

Clearly, design Q happens to be better than all other designs in terms of maximum number of shorter path-length, hence it becomes the best design to be used for experimentation when connectedness is the focus.

of-fit for these distributions has been compared and presented in the table and garaphically in figure 3.

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