Stability Analysis of Some Fixed Point Iterative Procedures

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***Abstract*— Several approaches have been used to obtain results on the stability of S-Iteration, Thiawan iteration and Picard-Mann iteration when dealing with different classes of quasi-contractive operators. In this paper, we established the stability analysis of S-Iteration, Thiawan iteration and Picard-Mann iteration using Simeon Riech contractive condition. Moreover, the aforementioned iterative schemes were shown to be T- stable.**

Keywords: *Simeon Riech contractive condition; T-Stable; S-Iterative procedure; Thianwan Iterative Procedure and Picard-Mann Iterative procedure.*

I. Introduction

Let X be a normed linear space and T is a function mapping X to itself. Suppose $x_0 \in X$ and $x_{n+1} = f(T, x_n)$ are iterative procedures which yields a sequence of points $\{x_n\}$ in X. Let $F(T) = \{x \in X: Tx = x\} \neq \emptyset$ and that $\{x_n\}_{n=0}^{\infty}$ converges strongly to $p \in F(T)$. Suppose $\{y_n\}_{n=0}^{\infty}$ is a sequence in X and $\{\epsilon_n\}$ is a sequence in $[0, \infty)$ given by $\epsilon_n = \|y_{n+1} - f(T, y_n)\|$.

If $\lim_{n\to\infty} \epsilon_n$ implies $\lim_{n\to\infty} y_n = p$, then the iteration procedure defined by

$$x_{n+1} = f(T, x_n)$$

is said to be T- stable or stable with respect to T. If $\sum_{n=0}^{\infty} \epsilon_n < \infty$ implies $y_n \to p$, then the iteration procedure is said to be almost T −stable. Clearly, any T-stable iteration procedure is almost T −stable, but the converse may not necessarily be true.See [1-3, 5, 6 & 10].

II. Research Methodology

Let (X, d) be a metric space, T a self- map of X with $F_T = \{x \in X: = x\} \neq \emptyset$ and consider a fixed point iteration procedure, that is, a sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_0 \in X$ and

$$x_{n+1} = f(T, x_n), \ n = 1, 2, 3, \cdots \tag{1}$$

where f is a function.

Definition. 1

(i) S-Iteration is defined as:

$$f(T, x_n) = (1 - \alpha_n)Ty_n + \alpha_n Ty_n,$$

where $y_n = (1 - \beta_n)x_n + \beta_n Tx_n$,

$\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty} \subset [0, 1]$ (2)

(ii) Thaiwan defined the following iteration:

$$f(T, x_n) = (1 - \alpha_n)y_n + \alpha_n Ty_n,$$

where $y_n = (1 - \beta_n)x_n + \beta_n Tx_n$,

$\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty} \subset [0, 1]$ (3)

(iii) Picard-Mann iteration is defined as

$$f(T, x_n) = Ty_n$$

where $\boldsymbol{y_n = (1 - \alpha_n)x_n + \alpha_n T x_n}$ *and*
$\{\alpha_n\}_{n=0}^{\infty} \subset [0, 1]$ (4)

Definition 2 (Simeon Riech Contraction mapping)
Let T be a complete metric space with distance function d and T is a function mapping X into itself, the following contractive type of mapping holds:
If there exists non-negative numbers a, b, c satisfying $a + b + c < 1$ such that for each $x, y \in X$, we have

$$d(Tx, Ty) \leq ad(x, T(x)) + bd(y, T(y)) + cd(x, y) \quad (5)$$

Lemma 1
Let $\{a_n\}_{n=0}^{\infty}$, and $\{b_n\}_{n=0}^{\infty}$ be sequences of non-negative numbers and $0 \leq q < 1$ so that $a_{n+1} \leq qa_n + b_n$ for all $n \geq 0$:

(i) if $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n = 0$;
(ii) if $\sum_{n=0}^{\infty} b_n < \infty$, then $\sum_{n=0}^{\infty} a_n < \infty$.

Remark If $q = 1$, then the above result holds in a weaker form. See [4, 7, 8 & 9].

III. Analysis

The analysis of S-Iteration, Thiawan iteration and Picard-Mann iteration via Simeon Riech contractive condition.

IV. Results

In this section, we shall prove the stability analysis of S-Iteration, Thiawan iteration and Picard-Mann iteration using Simeon Riech contractive condition.

Theorem 1
Let X be a normed linear space and $T: X \to X$ be a mapping satisfying (5) with $(d(u, v) = \|u - v\|)$. Suppose T has a fixed point p. Let x_0 be arbitrary element but fixed in X and define $\{x_n\}_{n=0}^{\infty}$ as (2), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1] such that $0 < \propto, \beta \leq \propto_n \beta_n$ for some α, β. Let $\{y_n\}$ be any sequence in X with $\epsilon_n = y_{n+1} - ((1 - \alpha_n)TS_n + \alpha_n TS_n)$ and $S_n = (1 - \beta_n)y_n + \beta_n TS_n$. Then, $\{x_n\}$ converges strongly to p and is T-stable with respect to T.

Proof:
Let $f(T, x_n) = (1 - \alpha_n)Ty_n + \alpha_n Ty_n$, where $y_n = (1 - \beta_n)x_n + \beta_n Tx_n$

implies $\epsilon_n = \|y_{n+1} - f(T, x_n)\|$. Therefore,

$$\|y_{n+1} - p\| = \|y_{n+1} - ((1 - \alpha_n)TS_n + \alpha_n TS_n)\| + \|((1 - \alpha_n)TS_n + \alpha_n TS_n) - p\|$$

$$\leq \epsilon_n + \|((1 - \alpha_n)TS_n) - p\| + \|(\alpha_n TS_n) - p\|$$

$$= \epsilon_n + (1 - \alpha_n)\|TS_n - p\| + \alpha_n\|TS_n - p\|$$

$$= \epsilon_n + [(1 - \alpha_n) + \alpha_n]\|TS_n - p\|$$

by (5), we have

$$\|y_{n+1} - p\| \leq \epsilon_n + [(1 - \alpha_n) + \alpha_n][a\|S_n - p\| + b\|S_n - TS_n\| + c\|p - Tp\|]$$

$$\leq \epsilon_n + [(1 - \alpha_n) + \alpha_n]\, a\|S_n - p\|$$

$$\leq a\|S_n - p\| + \epsilon_n.$$

Also,

$$S_n = (1 - \beta_n)y_n + \beta_n TS_n$$

$$\leq a\|[(1 - \beta_n)y_n + \beta_n Ty_n] - p\| + \epsilon_n$$

$$\leq a\|[(1 - \beta_n)y_n] - p\| + \|[\beta_n Ty_n] - p\| + \epsilon_n$$

$\leq a(1-\beta_n)\|y_n - p\| + a\beta_n\|Ty_n - p\| + \epsilon_n$

by applying (5) we have

$S_n \leq a(1-\beta_n)\|y_n - p\| + a\beta_n[a\|y_n - p\| + b\|y_n - Ty_n\| + c\|p - Tp\|] + \epsilon_n$

$\leq a(1-\beta_n)\|y_n - p\| + a^2\beta_n\|y_n - p\| + \epsilon_n$

$\leq [a(1-\beta_n) + a^2\beta_n]\ \|y_n - p\| + \epsilon_n$

by Lemma 1 and the fact that $a+b+c<1$, $\sum_{n=1}^{\infty}\alpha_n, \beta_n = \infty$, $\lim_{n\to 0}\epsilon_n = 0$ and $\lim_{n\to 0}\|y_n - p\| = 0$ implies $\|y_n - p\| = 0$ and $y_n = p$.

But $y_n \approx x_n$, therefore $x_n = p$. ■

Theorem 2

Let X be a normed linear space and $T: X \to X$ be a mapping satisfying (5) with $(d(u,v) = \|u-v\|)$. Suppose T has a fixed point p. Let x_0 be arbitrary element but fixed in X and define $\{x_n\}_{n=0}^{\infty}$ as (3) where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1] such that $0 < \alpha, \beta \leq \alpha_n\beta_n$ for some α, β. Let $\{y_n\}$ be any sequence in X with

$\epsilon_n = \|y_{n+1} - ((1-\alpha_n)S_n + \alpha_n TS_n)\|$ and $S_n = (1-\beta_n)y_n + \beta_n TS_n$.

Then, $\{x_n\}$ converges strongly to p and is T-stable with respect to T.

Proof:

Suppose $f(T, x_n) = (1-\alpha_n)y_n + \alpha_n Ty_n$ where $y_n = (1-\beta_n)x_n + \beta_n Tx_n$ that is $\epsilon_n = \|y_{n+1} - f(T, x_n)\|$. Hence,

$\|y_{n+1} - p\| = \|y_{n+1} - ((1-\alpha_n)S_n + \alpha_n TS_n)\| + \|((1-\alpha_n)S_n + \alpha_n TS_n) - p\|$

$\leq \epsilon_n + \|((1-\alpha_n)S_n) - p\| + \|(\alpha_n TS_n) - p\|$

$=(1-\alpha_n)\|S_n - p\| + \alpha_n\|TS_n - p\| + \epsilon_n$

by applying (5), we have

$\|y_{n+1} - p\| \leq [(1-\alpha_n)\|S_n - p\| + \alpha_n][a\|S_n - p\| + b\|S_n - TS_n\| + c\|p - Tp\|] + \epsilon_n$

$\leq (1-\alpha_n)\|S_n - p\| + \alpha_n a\|S_n - p\| + \epsilon_n$

$\leq (1-\alpha_n + \alpha_n a)\|S_n - p\| + \epsilon_n$

by definition

$S_n = (1-\beta_n)y_n + \beta_n TS_n$

$\leq (1-\alpha_n + \alpha_n a)\|[(1-\beta_n)y_n + \beta_n Ty_n] - p\| + \epsilon_n$

$\leq (1-\alpha_n + \alpha_n a)\|[(1-\beta_n)y_n] - p\| + \|[\beta_n Ty_n] - p\| + \epsilon_n$

$\leq (1-\alpha_n + \alpha_n a)(1-\beta_n)\|y_n - p\| + \beta_n\|Ty_n - p\| + \epsilon_n$

by (5), we have S_n

$\leq (1-\alpha_n + \alpha_n a)(1-\beta_n)\|y_n - p\| + \beta_n[a\|y_n - p\| + b\|y_n - Ty_n\| + c\|p - Tp\|] + \epsilon_n$

$\leq (1-\alpha_n + \alpha_n a)(1-\beta_n)\|y_n - p\| + a\beta_n\|y_n - p\| + \epsilon_n$

$$\leq (1-\alpha_n+\alpha_n a)(1-\beta_n+a\beta_n)\|y_n - p\| + \epsilon_n$$

by Lemma 1 and the fact that $a+b+c<1$, $\sum_{n=1}^{\infty}\alpha_n,\beta_n=\infty$.

$\lim_{n\to 0}\epsilon_n = 0$ and $\lim_{n\to 0}\|y_n - p\| = 0$, implies $\|y_n - p\| = 0$. Hence, $y_n = p$
but $y_n \approx x_n$, therefore, $x_n = p$. ■

Theorem 3

Let X be a normed linear space and $T: X \to X$ be a mapping satisfying (5) with $(d(u,v) = \|u-v\|)$. Suppose T has a fixed point p. Let x_0 be arbitrary element but fixed in X and define $\{x_n\}_{n=0}^{\infty}$ as (4), where $\{\alpha_n\}$ is a sequences in [0,1] such that $0<\alpha,\beta \leq \alpha_n\beta_n$ for some α,β. Let $\{y_n\}$ be any sequence in X satisfying $\epsilon_n = \|y_{n+1} - TS_n\|$and $S_n = (1-\alpha_n)y_n + Ty_n$. Then, $\{x_n\}$ converges strongly to p and is T-stable.

Proof:

Let $f(T,x_n) = Ty_n$ where, $y_n = (1-\alpha_n)x_n + Tx_n$ then $\epsilon_n = \|y_{n+1} - f(T,x_n)\|$

therefore,

$$\|y_{n+1} - p\| = \|y_{n+1} - TS_n)\| + \|TS_n - p\| \leq \epsilon_n + \|TS_n - p\|$$

$$\leq [a\|S_n - p\| + b\|S_n - TS_n\| + c\|p - Tp\|] + \epsilon_n$$

$$\leq a\|S_n - p\| + \epsilon_n$$

hence

$$S_n \leq a\|[(1-\alpha_n)y_n + Ty_n] - p\| + \epsilon_n$$

$$\leq a\|[(1-\alpha_n)y_n] - p\| + \|Ty_n - p\| + \epsilon_n$$

$$\leq a(1-\alpha_n)\|y_n - p\| + a\|Ty_n - p\| + \epsilon_n$$

$$\leq a(1-\alpha_n)\|y_n - p\| + a[a\|y_n - p\| + b\|y_n - Ty_n\| + c\|p - Tp\|] + \epsilon_n$$

$$\leq a(1-\alpha_n)\|y_n - p\| + a^2\|y_n - p\| + \epsilon_n$$

$$\leq [a(1-\alpha_n) + a^2]\|y_n - p\| + \epsilon_n$$

applying Lemma 1 and the fact that $a+b+c<1$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\lim_{n\to 0}\epsilon_n = 0$, $\lim_{n\to 0}\|y_n - p\| = 0$, implies, $\|y_n - p\| = 0$ and $y_n = p$. But, $y_n \approx x_n$, therefore, $x_n = p$. ■.

V. Discussions

The stability analysis of S-Iteration, Thiawan iteration and Picard-Mann iteration were shown and proved by means of Simeon Riech contractive condition.

VI. Conclusion

The result in this work obviously generalizes the results of several authors.

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