Comparison of Different Estimation Methods for the Half Logistic (Type I) Skew-t Distribution: a Monte Carlo Study

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***Abstract* - In the work, the techniques for determining the parameters of the type I half logistic skew-t ($TIHL_{ST}$) model using some classical estimation procedures is investigated. The six estimators considered are the maximum likelihood, least squares, weighted least squares, Anderson-Darling, maximum product of spacing and Cramer-von Mises. The finite sample performance of the $TIHL_{ST}$ parameters estimates are evaluated using the Monte Carlo simulations. The performances of all the six estimators were ordered based on ranks. The results showed that the performance ordering of the six estimators from the best to poorest using the overall ranks is the maximum likelihood followed by the Anderson-Darling, weighted least squares, maximum product of spacing, least squares, and Cramer-von Mises estimators in that order. The study concludes that the best method for estimating the parameters of type I half logistic skew-t model is the method of maximum likelihood.**

Keywords: *Maximum likelihood, Cramer-von Mises, Skew-t distribution, Type I half-logistic-G family, Monte Carlo.*

I. Introduction

The skew-t distribution with its complex form, lack of moment generating function and having a heavy-tail feature, has attracted many statisticians to propose more flexible and hybridized forms of it. These hybridized forms are capable of modeling real-life datasets with increasing, decreasing, bathtub, inverted bathtub, increasing-decreasing, and unimodal failure rates which are very common in various fields such as econometric, medical, time series, reliability, finance and volatility analysis. Some prominent hybridized forms of the skew-t model are, the exponentiated generalized skew-t (Dikko and Agboola, 2017), Balakrishnan skew-t distribution (Shafiei and Doostparast, 2014), generalized hyperbolic skew-t distribution (Aas and Haff, 2006), Kumaraswamy skew-t distribution (Khamis *et al.,* 2017), Beta skew-t distribution (Shittu *et al.,* 2014) and Beta skew-t distribution (Basalamah *et al.,* 2018), among many others.

Adubisi *et al.* (2020) proposed a hybridized form of the skew-t distribution by Jones and Faddy (2003) using the family of distributions introduced by (Cordeiro *et al.*, 2015). They studied the two-parameter type-I half logistic skew-t ($TIHL_{ST}$) distribution which can exhibit increasing, decreasing, or inverted bathtub hazard rate shapes. The $TIHL_{ST}$ density function can be symmetric, left-skewed and right-skewed. They also examined some of its important statistical properties such as the series expansion forms, quantile function, ordinary and incomplete moments, probability weighted moment, entropies and order statistics.

The $TIHL_{ST}$ distribution was applied to two real datasets from the medical and engineering fields and the results showed that it provided better fits to both datasets than the half logistic skew-t, basic pereto, exponentiated generalized pereto, Fréchet and skew-t distributions. Recently, there has been a key interest in the comparison of different classical estimation procedures for the parameter estimation of numerous distributions. For instance, the extended exponential geometric by (Louzada *et al.,* 2016), Poisson exponential by (Rodrigues *et al.,* 2018), Binomial exponential 2 by (Bakouch *et al.,* 2017), polynomial exponential by (Chesneau *et al.,* 2020), Fréchet by (Ramos *et al.,* 2020), odd exponential half-logistic exponential by (Aldahlan and Afify, 2020), type-I half-logistic Top-Leone by (ZeinEldin *et al.,* 2019), maximum likelihood estimation for the proportion difference of two-sample Binomial data by (Dewi *et al.,* 2019), among others.

The $TIHL_{ST}$ is created based on the type-I half logistic-G (TIHL-G) family of distributions introduced by (Cordeiro *et al*., 2015). The cumulative distribution function (CDF) is given by:

$$F(y;\varphi,\kappa)=\frac{\left(1-\left[1-G(y;\kappa)\right]^{\varphi}\right)}{\left(1+\left[1-G(y;\kappa)\right]^{\varphi}\right)},\quad \varphi>0, x\in\Re \qquad (1)$$

The probability density function (PDF) takes the form:

$$f(y;\varphi,\kappa)=\frac{2\varphi g(y;\kappa)\left\langle\left[1-G(y;\kappa)\right]^{\varphi-1}\right\rangle}{\left\{1+\left[1-G(y;\kappa)\right]^{\varphi}\right\}^{2}},\quad \varphi>0, x\in\Re \qquad (2)$$

where $G(y;\kappa)$ and $g(y;\kappa)$ are the respective baseline CDF and PDF depending on the parameter (κ) vector and $\varphi>0$ is the shape parameter, which gives more flexibility to the generated model to accommodate all important hazard rate function (HRF) shapes.

The CDF and PDF of the $TIHL_{ST}$ distribution (Adubisi et al., 2020) are given by

$$F(y,\upsilon)=\left\langle\frac{1-\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^{2}}}\right)\right)\right]^{\varphi}}{1+\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^{2}}}\right)\right)\right]^{\varphi}}\right\rangle,\quad y\in(-\infty,\infty),\varphi,\eta>0 \qquad (3)$$

and

$$f(y,\upsilon)=\frac{2\varphi\left(\frac{\eta}{2\left(\eta+y^{2}\right)^{3/2}}\right)\left\langle\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^{2}}}\right)\right)\right]^{\varphi-1}\right\rangle}{\left\{1+\left[1-\left(\frac{1}{2}\left(1+\frac{y}{\sqrt{\eta+y^{2}}}\right)\right)\right]^{\varphi}\right\}^{2}},\quad y\in(-\infty,\infty),\varphi,\eta>0 \qquad (4)$$

The $TIHL_{ST}$ distribution quantile function takes the form

$$x_{q}=\eta^{\frac{1}{2}}\frac{\left[1-2\left(\frac{1-u}{1+u}\right)^{\frac{1}{\varphi}}\right]}{\left\{\left[1-\left(1-2\left(\frac{1-u}{1+u}\right)^{\frac{1}{\varphi}}\right)^{2}\right]^{\frac{1}{2}}\right\}},\qquad 0<q<1. \qquad (5)$$

Adubisi *et al.* (2020) considered several statistical properties of the $TIHL_{ST}$ distribution in explicit forms including quantile function, ordinary and incomplete moments, probability weighted moment, entropies and order statistics. They also demonstrated its efficacy by analysing two real datasets from the engineering and medical fields. They only used the maximum likelihood method to estimate the $TIHL_{ST}$ parameters as the most common estimation method. Some shapes of the $TIHL_{ST}$ density and hazard rate functions are shown in Figures 1 and 2.

The aim of this article is to estimate the parameters of the $TIHL_{ST}$ distribution using different frequentist estimation methods such as the maximum likelihood estimation, least squares estimation, weighted least squares estimation, Anderson-Darling estimation, maximum product of spacing estimation and Cramer-von mises estimation. These estimation procedures using extensive Monte Carlo simulation are compared to discourse their performance.

This article is structured as follows: In section 2, the six classical estimators derived for the $TIHL_{ST}$ model parameters are presented. Section 3; extensive Monte Carlo simulation to compare the estimation procedures is performance. Conclusion in section 4.

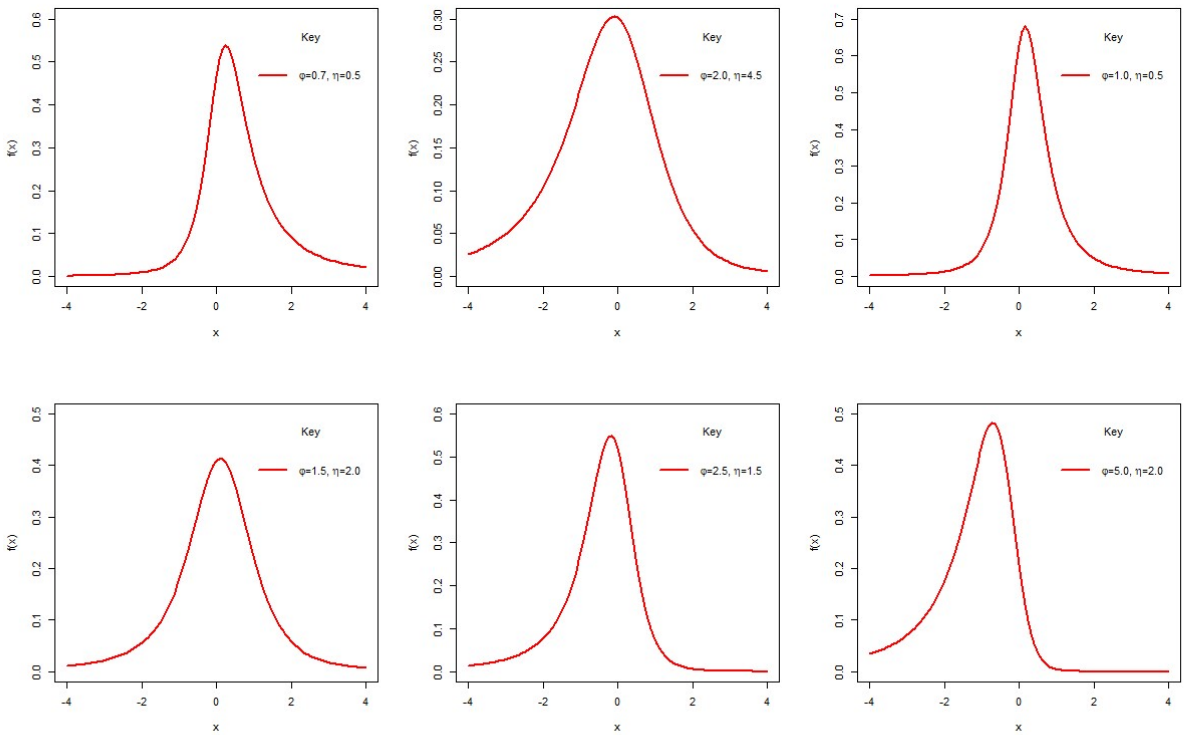

Figure 1: The density plots of the $TIHL_{ST}$ distribution for some chosen parameter values.

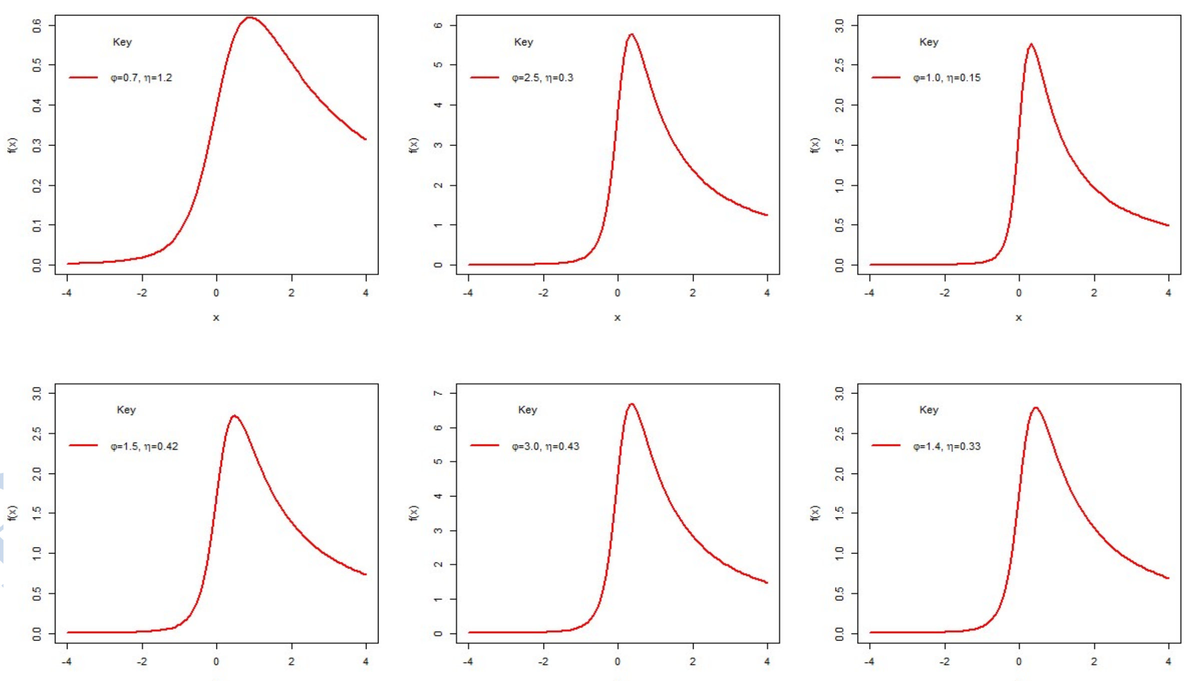

Figure 2: The hazard rate function plots of the $TIHL_{ST}$ distribution for some chosen parameter values.

II. Estimation Methods

This section discusses the estimation of the $TIHL_{ST}$ parameters φ and η, via six frequentist estimation methods. These estimation methods are the maximum likelihood, least squares, weighted least squares, Anderson-Darling, maximum product of spacing and Cramer-von mises methods.

2.1. Maximum Likelihood Estimation Method

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from the $TIHL_{ST}$ distribution with unknown parameter vector $\upsilon = (\varphi, \eta)^T$. The log-likelihood function, say l, is given as:

$$l = \log L(\upsilon) = n\log 2\varphi + n\log\eta - n\log 2 - 3/2\sum_{i=1}^{n}\log\left(\eta + y^2\right) + (\varphi - 1)\sum_{i=1}^{n}\log\left(1 - \frac{1}{2}\left(1 + \frac{y}{\sqrt{\eta + y^2}}\right)\right)$$
$$-2\sum_{i=1}^{n}\log\left(1 + \left(1 - \frac{1}{2}\left(1 + \frac{y}{\sqrt{\eta + y^2}}\right)\right)^{\varphi}\right) \quad (6)$$

The partial derivative of the log-likelihood l, with respect to φ and η equating to zero, the following non-linear equations are obtained as follows:

$$\frac{\partial l}{\partial \varphi} = \frac{n}{\varphi} + \sum_{i=1}^{n}\ln\left(1 - \frac{1}{2}\left(1 + \frac{y}{\sqrt{\eta + y^2}}\right)\right) - 2\sum_{i=1}^{n}\frac{\left(1 - \frac{1}{2}\left(1 + \frac{y}{\sqrt{\eta + y^2}}\right)\right)^{\varphi}\ln\left(1 - \frac{1}{2}\left(1 + \frac{y}{\sqrt{\eta + y^2}}\right)\right)}{\left\{1 + \left(1 - \frac{1}{2}\left(1 + \frac{y}{\sqrt{\varphi + y^2}}\right)\right)^{\alpha}\right\}} = 0 \quad (7)$$

$$\frac{\partial l}{\partial \eta} = \frac{n}{\eta} - \frac{3}{2}\sum_{i=1}^{n}\frac{1}{\left(\eta + y^2\right)} + (\varphi - 1)\sum_{i=1}^{n}\frac{y}{4\left(\eta + y^2\right)^{3/2}\left(1 - \frac{1}{2}\left(1 + \frac{y}{\sqrt{\eta + y^2}}\right)\right)}$$
$$-\varphi\sum_{i=1}^{n}\frac{x\left(1 - \frac{1}{2}\left(1 + \frac{y}{\sqrt{\eta + y^2}}\right)\right)^{\varphi}}{2\left(\eta + y^2\right)^{3/2}\left(1 - \frac{1}{2}\left(1 + \frac{y}{\sqrt{\eta + y^2}}\right)\right)\left\{1 + \left(1 - \frac{1}{2}\left(1 + \frac{y}{\sqrt{\eta + y^2}}\right)\right)^{\alpha}\right\}} = 0 \quad (8)$$

The nonlinear system of equations can be solved numerically via iterative methods using statistical programs (R, MATLAB, SAS, Mathematica and Maple), due to its complexity. In this work, R-software was used.

2.2 The Maximum Product of Spacing Estimation Method

The maximum product of spacing estimation (MPS) method proposed by (Cheng and Amin, 1979; Cheng and Amin, 1983) and developed by (Ranneby, 1984), is a good substitute to the maximum likelihood estimator (MLE) for the estimation of continuous univariate probability distribution unknown parameters. Let $y_{(1:n)}, y_{(2:n)}, \ldots, y_{(n:n)}$ be the ordered sample from the $TIHL_{ST}$ distribution and the uniform spacing for this random sample is given by:

$$D_i(\varphi, \eta) = F\left(y_{(i)} \middle| \varphi, \eta\right) - F\left(y_{(i-1)} \middle| \varphi, \eta\right), \quad \text{for } i = 1, 2, \ldots, n+1, \quad (9)$$

where $F\left(y_{(0)} \middle| \varphi, \eta\right) = 0$, $F\left(y_{(n+1)} \middle| \varphi, \eta\right) = 1$, such that $\sum_{i=1}^{n+1} D_i(\varphi, \eta) = 1$. Then,

$$F\left(y_{(i)}|\varphi,\eta\right)=\left\{\frac{1-\left[1-\left(\frac{1}{2}\left(1+\frac{y_{(i)}}{\sqrt{\eta+y_{(i)}^2}}\right)\right)\right]^{\varphi}}{1+\left[1-\left(\frac{1}{2}\left(1+\frac{y_{(i)}}{\sqrt{\eta+y_{(i)}^2}}\right)\right)\right]^{\varphi}}\right\}$$

and

$$F\left(y_{(i-1)}|\varphi,\eta\right)=\left\{\frac{1-\left[1-\left(\frac{1}{2}\left(1+\frac{y_{(i-1)}}{\sqrt{\eta+y_{(i-1)}^2}}\right)\right)\right]^{\varphi}}{1+\left[1-\left(\frac{1}{2}\left(1+\frac{y_{(i-1)}}{\sqrt{\eta+y_{(i-1)}^2}}\right)\right)\right]^{\varphi}}\right\}.$$

The maximum product of spacing estimates (MPSEs) of $\hat{\varphi}_{MPS}$ and $\hat{\eta}_{MPS}$ are obtained by maximising either the geometric mean of spacings or the logarithm of the geometric mean of sample spacings with respect to φ, and η

$$G(\varphi,\eta)=\left\langle\prod_{i=1}^{n+1}D_i(\varphi,\eta)\right\rangle^{\frac{1}{n+1}} \quad \text{and}$$

$$H(\varphi,\eta)=\frac{1}{n+1}\sum_{i=1}^{n+1}\log\left\langle D_i(\varphi,\eta)\right\rangle \tag{10}$$

The MPS estimates of the EHL$_{ST}$ parameters can be obtained by solving the following nonlinear equations

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\varphi,\eta)}\left\langle\Delta_t\left(y_{(i)}|\varphi,\eta\right)-\Delta_t\left(y_{(i-1)}|\varphi,\eta\right)\right\rangle=0, \quad \text{for } t=1,2 \tag{11}$$

where

$$\Delta_1\left(y_{(i)}|\varphi,\eta\right)=\frac{\partial}{\partial\varphi}F\left(y_{(i)}|\varphi,\eta\right)$$

$$\Delta_3\left(y_{(i)}|\varphi,\eta\right)=\frac{\partial}{\partial\eta}F\left(y_{(i)}|\varphi,\eta\right) \tag{12}$$

It is important to mention that Δ_t, for $t=1,2$ can be obtained numerically.

2.3 The Anderson-Darling Estimation Method

The Anderson-Darling estimation (ANDE) method is a type of minimum distance estimator used in estimating the parameters of a continuous univariate distribution based on the Anderson-Darling statistic (Lucceno, 2006). The Anderson-Darling estimates of the TIHL$_{ST}$ parameters can be obtained by minimizing with respect to φ and η, the function given by:

$$C(\varphi,\eta)=\frac{1}{12n}+\sum_{i=1}^{n}\left\langle\left\{\frac{2i-1}{2n}\right\}-F\left(y_{(i:n)}|\varphi,\eta\right)\right\rangle^2, \tag{15}$$

Solving the following nonlinear equations, the Cramer-von Mises estimates can also be obtained as follows:

$$\sum_{i=1}^{n}\left\langle F\left(y_{(i:n)}|\varphi,\eta\right)-\left\{\frac{2i-1}{2n}\right\}\right\rangle\Delta_t\left(y_{(i:n)}|\varphi,\eta\right)=0, \text{ for } t=1,2 \tag{16}$$

where $\Delta_1\left(\cdot|\varphi,\eta\right)$ and $\Delta_2\left(\cdot|\varphi,\eta\right)$ are given in (12).

2.4 The Cramer-von Mises Estimation Method

The Cramer-von Mises estimation (CVME) method is considered a type of minimum distance estimator that have smaller bias than other minimum distance estimators [21]. The Cramer-von Mises estimator (CVME)is based on the difference between the estimates of the CDF and the empirical CDF (Lucceno, 2006; MacDonald, 1971). The Cramer-von Mises estimates of the EHL$_{ST}$ parameters can be obtained by minimizing with respect to φ and η, the function given by

$$C(\varphi,\eta)=\frac{1}{12n}+\sum_{i=1}^{n}\left\langle\left\{\frac{2i-1}{2n}\right\}-F\left(y_{(i:n)}|\varphi,\eta\right)\right\rangle^2, \tag{15}$$

Solving the following nonlinear equations, the Cramer-von Mises estimates can also be obtained as follows:

$$\sum_{i=1}^{n}\left\langle F\left(y_{(i:n)}|\varphi,\eta\right)-\left\{\frac{2i-1}{2n}\right\}\right\rangle\Delta_t\left(y_{(i:n)}|\varphi,\eta\right)=0, \text{ for } t=1,2 \tag{16}$$

where $\Delta_1\left(\cdot|\varphi,\eta\right)$ and $\Delta_2\left(\cdot|\varphi,\eta\right)$ are given in (12).

2.5 The Least Squares and Weighted Least Squares Estimation Methods

The least squares and weighted least squares estimations methods was proposed by (Swain *et al.,* 1988). Let

$y_{(1:n)}, y_{(2:n)}, \ldots, y_{(n:n)}$ be the ordered sample from the TIHL$_{ST}$ distribution. The least squares estimators (LSEs) of the TIHL$_{ST}$ parameters $\hat{\varphi}_{LSE}$ and $\hat{\eta}_{LSE}$ can be obtained by minimizing the least square function given by:

$$L(\varphi,\eta) = \sum_{i=1}^{n} \left\langle F\left(y_{(i:n)} | \varphi,\eta\right) - \left\{\frac{i}{n+1}\right\}\right\rangle^2 \quad (17)$$

with respect to φ and η. Likewise, the least squares estimators (LSEs) can be determined by solving the following non-linear equations:

$$\sum_{i=1}^{n} \left\langle F\left(y_{(i:n)} | \varphi,\eta\right) - \left\{\frac{i}{n+1}\right\}\right\rangle \Delta_t\left(y_{(i:n)} | \varphi,\eta\right) = 0, \text{ for } t = 1,2 \quad (18)$$

where $\Delta_1(\cdot|\varphi,\eta)$ and $\Delta_2(\cdot|\varphi,\eta)$ are given in (12).

The weighted least squares estimators (WLSEs) of the TIHL$_{ST}$ parameters $\hat{\varphi}_{LSE}$ and $\hat{\eta}_{LSE}$ can be obtained by minimizing, with respect to φ and η, the following equation:

$$W(\varphi,\eta) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left\langle F\left(y_{(i:n)} | \varphi,\eta\right) - \left\{\frac{i}{n+1}\right\}\right\rangle^2 \quad (19)$$

Likewise, the weighted least squares estimators (WLSEs) can be obtained by solving the following non-linear equations:

$$\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left\langle F\left(y_{(i:n)} | \varphi,\eta\right) - \left\{\frac{i}{n+1}\right\}\right\rangle \Delta_t\left(y_{(i:n)} | \varphi,\eta\right) = 0,$$

for $t = 1,2$ (20)

where $\Delta_1(\cdot|\varphi,\eta)$ and $\Delta_2(\cdot|\varphi,\eta)$ are given in (*).

III. Simulation Study

In this section, we compare and evaluate the performance of the six frequentist estimators of the TIHL$_{ST}$ parameters using Monte Carlo simulation study. We generated $N = 1000$ random sample of sizes $n = 50, 100, 300, 1000$ from the TIHL$_{ST}$ distribution. The parameter combinations (φ,η) are: $(\varphi=2,\eta=0.5), (\varphi=2,\eta=1.5), (\varphi=2,\eta=2), (\varphi=3,\eta=0.5), (\varphi=3,\eta=1.5)$ and $(\varphi=3,\eta=2)$. For each parameter combination, the MLE, MPSE, ANDE, CVME, LSE and WLSE of φ and η were determined for each sample, allowing the computation of the following measures: average estimates (AVEs), average absolute biases (AVABs) and mean square errors (MSEs) of these estimators. We also provide the partial and overall ranks of the six methods of estimation for various parameter combinations to determine the best estimation method for the TIHL$_{ST}$ parameters.

Tables 1 – 6 provides the AVEs, AVABs and MSEs of the six estimation methods. More so, these tables show the rank of each estimator among all the estimators in each row, the superscripts are the indicators and $\sum ranks$ is the partial sum of the ranks for each column and each sample size. Table 7 provides the partial and overall ranks of the six estimation methods for all the TIHL$_{ST}$ parameter combinations.

Table 1. Simulation results of the six estimation methods for $\varphi = 2, \eta = 0.5$.

n	Measures	Pa.	MLE	MPSE	ANDE	CVME	LSE	WLSE
50	AVE	φ	2.0374$^{(5)}$	1.9963$^{(1)}$	2.0306$^{(4)}$	2.0426$^{(6)}$	2.0285$^{(2)}$	2.0287$^{(3)}$
		η	0.5061$^{(2)}$	0.5564$^{(6)}$	0.5180$^{(3)}$	0.5036$^{(1)}$	0.5372$^{(5)}$	0.5245$^{(4)}$
	AVAB	φ	0.0374$^{(5)}$	0.0037$^{(1)}$	0.0306$^{(4)}$	0.0426$^{(6)}$	0.0285$^{(2)}$	0.0287$^{(3)}$
		η	0.0061$^{(2)}$	0.0564$^{(6)}$	0.0180$^{(3)}$	0.0036$^{(1)}$	0.0372$^{(5)}$	0.0245$^{(4)}$
	MSE	φ	0.0750$^{(2)}$	0.0676$^{(1)}$	0.0763$^{(3)}$	0.0892$^{(6)}$	0.0815$^{(5)}$	0.0775$^{(4)}$
		η	0.0265$^{(1)}$	0.0357$^{(6)}$	0.0281$^{(3)}$	0.0271$^{(2)}$	0.0320$^{(5)}$	0.0294$^{(4)}$
	$\sum ranks$		**17$^{(1)}$**	**21$^{(3)}$**	**20$^{(2)}$**	**22$^{(4.5)}$**	**24$^{(6)}$**	**22$^{(4.5)}$**
100	AVE	φ	2.0155$^{(5)}$	1.9907$^{(1)}$	2.0125$^{(4)}$	2.0181$^{(6)}$	2.0117$^{(3)}$	2.0116$^{(2)}$
		η	0.5073$^{(2)}$	0.5320$^{(6)}$	0.5129$^{(3)}$	0.5071$^{(1)}$	0.5236$^{(5)}$	0.5152$^{(4)}$
	AVAB	φ	0.0155$^{(5)}$	0.0093$^{(1)}$	0.0125$^{(4)}$	0.0181$^{(6)}$	0.0117$^{(3)}$	0.0116$^{(2)}$
		η	0.0073$^{(2)}$	0.0320$^{(6)}$	0.0129$^{(3)}$	0.0071$^{(1)}$	0.0236$^{(5)}$	0.0152$^{(4)}$
	MSE	φ	0.0339$^{(2)}$	0.0323$^{(1)}$	0.0357$^{(3)}$	0.0392$^{(6)}$	0.0376$^{(5)}$	0.0358$^{(4)}$
		η	0.0135$^{(1)}$	0.0160$^{(6)}$	0.0140$^{(2)}$	0.0145$^{(4)}$	0.0159$^{(5)}$	0.0142$^{(3)}$
	$\sum ranks$		**17$^{(1)}$**	**21$^{(4)}$**	**19$^{(2.5)}$**	**24$^{(5)}$**	**26$^{(6)}$**	**19$^{(2.5)}$**
300	AVE	φ	2.0031$^{(5)}$	1.9922$^{(1)}$	2.0022$^{(4)}$	2.0041$^{(6)}$	2.0021$^{(2.5)}$	2.0021$^{(2.5)}$
		η	0.4996$^{(2)}$	0.5074$^{(6)}$	0.5015$^{(3)}$	0.4995$^{(1)}$	0.5049$^{(5)}$	0.5016$^{(4)}$
	AVAB	φ	0.0031$^{(4)}$	0.0078$^{(6)}$	0.0022$^{(3)}$	0.0041$^{(5)}$	0.0021$^{(1.5)}$	0.0021$^{(1.5)}$
		η	0.0004$^{(1)}$	0.0074$^{(6)}$	0.0015$^{(3)}$	0.0005$^{(2)}$	0.0049$^{(5)}$	0.0016$^{(4)}$
	MSE	φ	0.0111$^{(2)}$	0.0110$^{(1)}$	0.0114$^{(3)}$	0.0123$^{(6)}$	0.0121$^{(5)}$	0.0115$^{(4)}$
		η	0.0039$^{(1)}$	0.0041$^{(4)}$	0.0040$^{(2.5)}$	0.0043$^{(5)}$	0.0044$^{(6)}$	0.0040$^{(2.5)}$
	$\sum ranks$		**15$^{(1)}$**	**24$^{(4)}$**	**18.5$^{(2.5)}$**	**25$^{(5.5)}$**	**25$^{(5.5)}$**	**18$^{(2.5)}$**
1000	AVE	φ	1.9992$^{(5)}$	1.9950$^{(1)}$	1.9989$^{(4)}$	1.9993$^{(6)}$	1.9987$^{(2)}$	1.9988$^{(3)}$
		η	0.4981$^{(1)}$	0.5004$^{(6)}$	0.4987$^{(3.5)}$	0.4985$^{(2)}$	0.5001$^{(5)}$	0.4987$^{(3.5)}$
	AVAB	φ	0.0008$^{(3)}$	0.0005$^{(1)}$	0.0011$^{(5)}$	0.0007$^{(2)}$	0.0013$^{(6)}$	0.0012$^{(5)}$
		η	0.0019$^{(6)}$	0.00004$^{(1)}$	0.0013$^{(3.5)}$	0.0015$^{(5)}$	0.0001$^{(2)}$	0.0013$^{(3.5)}$
	MSE	φ	0.0031$^{(1.5)}$	0.0031$^{(1.5)}$	0.0033$^{(3.5)}$	0.0035$^{(5.5)}$	0.0035$^{(5.5)}$	0.0033$^{(3.5)}$
		η	0.0012$^{(2.5)}$	0.0012$^{(2.5)}$	0.0012$^{(2.5)}$	0.0013$^{(5.5)}$	0.0013$^{(5.5)}$	0.0012$^{(2.5)}$
	$\sum ranks$		**19$^{(2)}$**	**13$^{(1)}$**	**22$^{(4)}$**	**26$^{(5.5)}$**	**26$^{(5.5)}$**	**21$^{(3)}$**

Table 2. Simulation results of the six estimation methods for $\varphi = 2, \eta = 1.5$.

n	Measures	Pa.	MLE	MPSE	ANDE	CVME	LSE	WLSE
50	AVE	φ	$2.0374^{(5)}$	$1.9963^{(1)}$	$2.0306^{(4)}$	$2.0426^{(6)}$	$2.0285^{(2)}$	$2.0287^{(3)}$
		η	$1.5185^{(2)}$	$1.6692^{(6)}$	$1.5539^{(3)}$	$1.5108^{(1)}$	$1.6117^{(5)}$	$1.5736^{(4)}$
	AVAB	φ	$0.0374^{(5)}$	$0.0037^{(1)}$	$0.0306^{(4)}$	$0.0426^{(6)}$	$0.0285^{(2)}$	$0.0287^{(3)}$
		η	$0.0185^{(2)}$	$0.1692^{(6)}$	$0.0539^{(3)}$	$0.0108^{(1)}$	$0.1117^{(5)}$	$0.0736^{(4)}$
	MSE	φ	$0.0750^{(2)}$	$0.0676^{(1)}$	$0.0763^{(3)}$	$0.0892^{(6)}$	$0.0815^{(5)}$	$0.0775^{(4)}$
		η	$0.2386^{(1)}$	$0.3217^{(6)}$	$0.2528^{(3)}$	$0.2441^{(2)}$	$0.2880^{(5)}$	$0.2648^{(4)}$
	$\sum ranks$		$\mathbf{17}^{(1)}$	$\mathbf{21}^{(3)}$	$\mathbf{20}^{(2)}$	$\mathbf{22}^{(4.5)}$	$\mathbf{24}^{(6)}$	$\mathbf{22}^{(4.5)}$
100	AVE	φ	$2.0154^{(5)}$	$1.9907^{(1)}$	$2.0125^{(4)}$	$2.0181^{(6)}$	$2.0117^{(3)}$	$2.0116^{(2)}$
		η	$1.5219^{(2)}$	$1.5965^{(6)}$	$1.5388^{(3)}$	$1.5212^{(1)}$	$1.5709^{(5)}$	$1.5456^{(4)}$
	AVAB	φ	$0.0154^{(5)}$	$0.0093^{(1)}$	$0.0125^{(4)}$	$0.0181^{(6)}$	$0.0117^{(3)}$	$0.0116^{(2)}$
		η	$0.0219^{(2)}$	$0.0965^{(6)}$	$0.0388^{(3)}$	$0.0212^{(1)}$	$0.0709^{(5)}$	$0.0456^{(4)}$
	MSE	φ	$0.0339^{(2)}$	$0.0323^{(1)}$	$0.0357^{(3)}$	$0.0392^{(6)}$	$0.0376^{(5)}$	$0.0358^{(4)}$
		η	$0.1218^{(1)}$	$0.1440^{(6)}$	$0.1262^{(2)}$	$0.1306^{(4)}$	$0.1431^{(5)}$	$0.1282^{(3)}$
	$\sum ranks$		$\mathbf{17}^{(1)}$	$\mathbf{21}^{(4)}$	$\mathbf{19}^{(2.5)}$	$\mathbf{24}^{(5)}$	$\mathbf{26}^{(6)}$	$\mathbf{19}^{(2.5)}$
300	AVE	φ	$2.0031^{(5)}$	$1.9922^{(1)}$	$2.0022^{(4)}$	$2.0041^{(6)}$	$2.0021^{(2.5)}$	$2.0021^{(2.5)}$
		η	$1.4988^{(2)}$	$1.5219^{(6)}$	$1.5044^{(3)}$	$1.4986^{(1)}$	$1.5147^{(5)}$	$1.5049^{(4)}$
	AVAB	φ	$0.0031^{(4)}$	$0.0077^{(6)}$	$0.0022^{(3)}$	$0.0041^{(5)}$	$0.0021^{(1.5)}$	$0.0021^{(1.5)}$
		η	$0.0012^{(1)}$	$0.0219^{(6)}$	$0.0044^{(3)}$	$0.0014^{(2)}$	$0.0147^{(5)}$	$0.0049^{(4)}$
	MSE	φ	$0.0111^{(2)}$	$0.0110^{(1)}$	$0.0114^{(3)}$	$0.0123^{(6)}$	$0.0121^{(5)}$	$0.0115^{(4)}$
		η	$0.0355^{(1)}$	$0.0371^{(4)}$	$0.0362^{(3)}$	$0.0390^{(5)}$	$0.0400^{(6)}$	$0.0361^{(2)}$
	$\sum ranks$		$\mathbf{15}^{(1)}$	$\mathbf{24}^{(4)}$	$\mathbf{19}^{(3)}$	$\mathbf{25}^{(5.5)}$	$\mathbf{25}^{(5.5)}$	$\mathbf{18}^{(2)}$
1000	AVE	φ	$1.9992^{(5)}$	$1.9950^{(1)}$	$1.9989^{(4)}$	$1.9993^{(6)}$	$1.9987^{(2)}$	$1.9988^{(3)}$
		η	$1.4943^{(1)}$	$1.5007^{(6)}$	$1.4962^{(4)}$	$1.4954^{(2)}$	$1.5002^{(5)}$	$1.4961^{(3)}$
	AVAB	φ	$0.0008^{(3)}$	$0.0005^{(1)}$	$0.0011^{(4)}$	$0.0007^{(2)}$	$0.0013^{(6)}$	$0.0012^{(5)}$
		η	$0.0057^{(6)}$	$0.00007^{(1)}$	$0.0038^{(3)}$	$0.0046^{(5)}$	$0.0002^{(2)}$	$0.0039^{(4)}$
	MSE	φ	$0.0031^{(1.5)}$	$0.0031^{(1.5)}$	$0.0033^{(3.5)}$	$0.0035^{(5.5)}$	$0.0035^{(5.5)}$	$0.0033^{(3.5)}$
		η	$0.0106^{(1)}$	$0.0107^{(3)}$	$0.0107^{(3)}$	$0.0117^{(5.5)}$	$0.0117^{(5.5)}$	$0.0107^{(3)}$
	$\sum ranks$		$\mathbf{17.5}^{(2)}$	$\mathbf{13.5}^{(1)}$	$\mathbf{21.5}^{(3.5)}$	$\mathbf{26}^{(5.5)}$	$\mathbf{26}^{(5.5)}$	$\mathbf{21.5}^{(3.5)}$

Table 3: Simulation results of the six estimation methods for $\varphi = 2, \eta = 2$.

n	Measures	Pa.	MLE	MPSE	ANDE	CVME	LSE	WLSE
50	AVE	φ	$2.0374^{(5)}$	$1.9963^{(1)}$	$2.0306^{(4)}$	$2.0426^{(6)}$	$2.0285^{(2)}$	$2.0287^{(3)}$
		η	$2.0247^{(2)}$	$2.2254^{(6)}$	$2.0719^{(3)}$	$2.0144^{(1)}$	$2.1490^{(5)}$	$2.0982^{(4)}$
	AVAB	φ	$0.0374^{(5)}$	$0.0037^{(1)}$	$0.0306^{(4)}$	$0.0426^{(6)}$	$0.0285^{(2)}$	$0.0287^{(3)}$
		η	$0.0247^{(2)}$	$0.2254^{(6)}$	$0.0719^{(3)}$	$0.0144^{(1)}$	$0.1490^{(5)}$	$0.0982^{(4)}$
	MSE	φ	$0.0750^{(2)}$	$0.0676^{(1)}$	$0.0763^{(3)}$	$0.0892^{(6)}$	$0.0815^{(5)}$	$0.0775^{(4)}$
		η	$0.4241^{(1)}$	$0.5713^{(6)}$	$0.4494^{(3)}$	$0.4339^{(2)}$	$0.5120^{(5)}$	$0.4707^{(4)}$
	$\sum ranks$		$\mathbf{17^{(1)}}$	$\mathbf{21^{(3)}}$	$\mathbf{20^{(2)}}$	$\mathbf{22^{(4.5)}}$	$\mathbf{24^{(6)}}$	$\mathbf{22^{(4.5)}}$
100	AVE	φ	$2.0154^{(5)}$	$1.9907^{(1)}$	$2.0125^{(4)}$	$2.0181^{(6)}$	$2.0117^{(3)}$	$2.0116^{(2)}$
		η	$2.0292^{(2)}$	$2.1285^{(6)}$	$2.0517^{(3)}$	$2.0282^{(1)}$	$2.0945^{(5)}$	$2.0608^{(4)}$
	AVAB	φ	$0.0154^{(5)}$	$0.0093^{(1)}$	$0.0125^{(4)}$	$0.0181^{(6)}$	$0.0117^{(3)}$	$0.0116^{(2)}$
		η	$0.0292^{(2)}$	$0.1285^{(6)}$	$0.0517^{(3)}$	$0.0282^{(1)}$	$0.0945^{(5)}$	$0.0608^{(4)}$
	MSE	φ	$0.0339^{(2)}$	$0.0323^{(1)}$	$0.0357^{(3)}$	$0.0392^{(6)}$	$0.0376^{(5)}$	$0.0358^{(4)}$
		η	$0.2162^{(1)}$	$0.2565^{(6)}$	$0.2243^{(2)}$	$0.2321^{(4)}$	$0.2545^{(5)}$	$0.2280^{(3)}$
	$\sum ranks$		$\mathbf{17^{(1)}}$	$\mathbf{21^{(4)}}$	$\mathbf{19^{(2.5)}}$	$\mathbf{24^{(5)}}$	$\mathbf{26^{(6)}}$	$\mathbf{19^{(2.5)}}$
300	AVE	φ	$2.0031^{(5)}$	$1.9922^{(1)}$	$2.0022^{(4)}$	$2.0041^{(6)}$	$2.0021^{(2.5)}$	$2.0021^{(2.5)}$
		η	$1.9985^{(2)}$	$2.0299^{(6)}$	$2.0058^{(3)}$	$1.9981^{(1)}$	$2.0196^{(5)}$	$2.0065^{(4)}$
	AVAB	φ	$0.0031^{(4)}$	$0.0078^{(6)}$	$0.0022^{(3)}$	$0.0041^{(5)}$	$0.0021^{(1.5)}$	$0.0021^{(1.5)}$
		η	$0.0015^{(1)}$	$0.0299^{(6)}$	$0.0058^{(3)}$	$0.0019^{(2)}$	$0.0196^{(5)}$	$0.0065^{(4)}$
	MSE	φ	$0.0111^{(2)}$	$0.0110^{(1)}$	$0.0114^{(3)}$	$0.0123^{(6)}$	$0.0121^{(5)}$	$0.0115^{(4)}$
		η	$0.0631^{(1)}$	$0.0665^{(4)}$	$0.0643^{(2.5)}$	$0.0693^{(5)}$	$0.0711^{(6)}$	$0.0643^{(2.5)}$
	$\sum ranks$		$\mathbf{15^{(1)}}$	$\mathbf{24^{(4)}}$	$\mathbf{18.5^{(2.5)}}$	$\mathbf{25^{(5.5)}}$	$\mathbf{25^{(5.5)}}$	$\mathbf{18.5^{(2.5)}}$
1000	AVE	φ	$1.9992^{(5)}$	$1.9951^{(1)}$	$1.9989^{(4)}$	$1.9993^{(6)}$	$1.9987^{(2)}$	$1.9988^{(3)}$
		η	$1.9924^{(1)}$	$2.0017^{(6)}$	$1.9949^{(4)}$	$1.9938^{(2)}$	$2.0002^{(5)}$	$1.9948^{(3)}$
	AVAB	φ	$0.0008^{(2)}$	$0.0049^{(6)}$	$0.0011^{(3)}$	$0.0007^{(1)}$	$0.0013^{(5)}$	$0.0012^{(4)}$
		η	$0.0076^{(6)}$	$0.0017^{(2)}$	$0.0051^{(3)}$	$0.0062^{(5)}$	$0.0002^{(1)}$	$0.0052^{(4)}$
	MSE	φ	$0.0031^{(1.5)}$	$0.0031^{(1.5)}$	$0.0033^{(3.5)}$	$0.0035^{(5.5)}$	$0.0035^{(5.5)}$	$0.0033^{(3.5)}$
		η	$0.0189^{(1)}$	$0.0191^{(3)}$	$0.0191^{(3)}$	$0.0207^{(5)}$	$0.0208^{(6)}$	$0.0191^{(3)}$
	$\sum ranks$		$\mathbf{16.5^{(1)}}$	$\mathbf{19.5^{(2)}}$	$\mathbf{20.5^{(3.5)}}$	$\mathbf{24.5^{(6)}}$	$\mathbf{22.5^{(5)}}$	$\mathbf{20.5^{(3.5)}}$

Table 4: Simulation results of the six estimation methods for $\varphi = 3, \eta = 0.5$.

n	Measures	Pa.	MLEs	MPSEs	ANDE	CVME	LSE	WLSE
50	AVE	φ	$3.0815^{(5)}$	$2.9776^{(1)}$	$3.0586^{(4)}$	$3.1025^{(6)}$	$3.0428^{(2)}$	$3.0500^{(3)}$
		η	$0.5028^{(1)}$	$0.5629^{(6)}$	$0.5190^{(3)}$	$0.5044^{(2)}$	$0.5379^{(5)}$	$0.5246^{(4)}$
	AVAB	φ	$0.0815^{(5)}$	$0.0224^{(1)}$	$0.0586^{(4)}$	$0.1025^{(6)}$	$0.0428^{(2)}$	$0.0500^{(3)}$
		η	$0.0028^{(1)}$	$0.0629^{(6)}$	$0.0190^{(3)}$	$0.0044^{(2)}$	$0.0379^{(5)}$	$0.0246^{(4)}$
	MSE	φ	$0.1673^{(3)}$	$0.1371^{(1)}$	$0.1629^{(2)}$	$0.2318^{(6)}$	$0.1944^{(5)}$	$0.1694^{(4)}$
		η	$0.0237^{(1)}$	$0.0338^{(6)}$	$0.0265^{(2)}$	$0.0273^{(3)}$	$0.0324^{(5)}$	$0.0280^{(4)}$
	$\sum ranks$		$\mathbf{16^{(1)}}$	$\mathbf{21^{(3)}}$	$\mathbf{18^{(2)}}$	$\mathbf{25^{(6)}}$	$\mathbf{24^{(5)}}$	$\mathbf{22^{(4)}}$
100	AVE	φ	$3.0318^{(5)}$	$2.9741^{(1)}$	$3.0218^{(2.5)}$	$3.0408^{(6)}$	$3.0128^{(2.5)}$	$3.0186^{(4)}$
		η	$0.5055^{(1)}$	$0.5372^{(6)}$	$0.5137^{(3)}$	$0.5079^{(2)}$	$0.5245^{(5)}$	$0.5153^{(4)}$
	AVAB	φ	$0.0318^{(5)}$	$0.0259^{(1)}$	$0.0218^{(1.5)}$	$0.0408^{(6)}$	$0.0128^{(1.5)}$	$0.0186^{(4)}$
		η	$0.0055^{(1)}$	$0.0372^{(6)}$	$0.0137^{(3)}$	$0.0079^{(2)}$	$0.0245^{(5)}$	$0.0153^{(4)}$
	MSE	φ	$0.0699^{(2)}$	$0.0637^{(1)}$	$0.0738^{(3)}$	$0.0909^{(6)}$	$0.0844^{(5)}$	$0.0756^{(4)}$
		η	$0.0118^{(1)}$	$0.0147^{(5)}$	$0.0130^{(2)}$	$0.0144^{(4)}$	$0.0158^{(6)}$	$0.0133^{(3)}$
	$\sum ranks$		$\mathbf{15^{(1.5)}}$	$\mathbf{20^{(3)}}$	$\mathbf{15^{(1.5)}}$	$\mathbf{26^{(6)}}$	$\mathbf{25^{(5)}}$	$\mathbf{23^{(4)}}$
300	AVE	φ	$3.0094^{(5)}$	$2.9866^{(1)}$	$3.0064^{(4)}$	$3.0129^{(6)}$	$3.0039^{(2)}$	$3.0061^{(3)}$
		η	$0.4994^{(1)}$	$0.5108^{(6)}$	$0.5019^{(4)}$	$0.4999^{(2)}$	$0.5053^{(5)}$	$0.5017^{(3)}$
	AVAB	φ	$0.0094^{(4)}$	$0.0134^{(6)}$	$0.0064^{(3)}$	$0.0129^{(5)}$	$0.0039^{(1)}$	$0.0061^{(2)}$
		η	$0.0006^{(2)}$	$0.0108^{(6)}$	$0.0019^{(4)}$	$0.0001^{(1)}$	$0.0053^{(5)}$	$0.0017^{(3)}$
	MSE	φ	$0.0223^{(2)}$	$0.0218^{(1)}$	$0.0233^{(3)}$	$0.0266^{(6)}$	$0.0260^{(5)}$	$0.0235^{(4)}$
		η	$0.0034^{(1)}$	$0.0037^{(4)}$	$0.0037^{(2.5)}$	$0.0043^{(5)}$	$0.0044^{(6)}$	$0.0037^{(2.5)}$
	$\sum ranks$		$\mathbf{15^{(1)}}$	$\mathbf{24^{(4.5)}}$	$\mathbf{20.5^{(3)}}$	$\mathbf{25^{(6)}}$	$\mathbf{24^{(4.5)}}$	$\mathbf{17.5^{(2)}}$
1000	AVE	φ	$3.0018^{(5)}$	$2.9940^{(1)}$	$3.0010^{(3.5)}$	$3.0026^{(6)}$	$2.9999^{(2)}$	$3.0010^{(3.5)}$
		η	$0.4981^{(1)}$	$0.5018^{(6)}$	$0.4989^{(4)}$	$0.4985^{(2)}$	$0.5002^{(5)}$	$0.4987^{(3)}$
	AVAB	φ	$0.0018^{(4)}$	$0.0060^{(6)}$	$0.0010^{(2.5)}$	$0.0026^{(5)}$	$0.00001^{(1)}$	$0.0010^{(2.5)}$
		η	$0.0019^{(6)}$	$0.0018^{(5)}$	$0.0011^{(2)}$	$0.0015^{(4)}$	$0.00002^{(1)}$	$0.0013^{(3)}$
	MSE	φ	$0.0063^{(2)}$	$0.0062^{(1)}$	$0.0067^{(3.5)}$	$0.0076^{(5)}$	$0.0075^{(6)}$	$0.0067^{(3.5)}$
		η	$0.0010^{(1.5)}$	$0.0010^{(1.5)}$	$0.0011^{(3.5)}$	$0.0013^{(5.5)}$	$0.0013^{(5.5)}$	$0.0011^{(3.5)}$
	$\sum ranks$		$\mathbf{19.5^{(3)}}$	$\mathbf{20.5^{(4.5)}}$	$\mathbf{19^{(1.5)}}$	$\mathbf{27.5^{(6)}}$	$\mathbf{20.5^{(4.5)}}$	$\mathbf{19^{(1.5)}}$

Table 5: Simulation results of the six estimation methods for $\varphi = 3, \eta = 1.5$.

n	Measures	Pa.	MLEs	MPSEs	ANDE	CVME	LSE	WLSE
50	AVE	φ	$3.0815^{(5)}$	$2.9775^{(1)}$	$3.0586^{(4)}$	$3.1025^{(6)}$	$3.0428^{(2)}$	$3.0500^{(3)}$
		η	$1.5083^{(1)}$	$1.6886^{(6)}$	$1.5571^{(3)}$	$1.5133^{(2)}$	$1.6137^{(5)}$	$1.5737^{(4)}$
	AVAB	φ	$0.0815^{(5)}$	$0.0225^{(1)}$	$0.0586^{(4)}$	$0.1025^{(6)}$	$0.0428^{(2)}$	$0.0500^{(3)}$
		η	$0.0083^{(1)}$	$0.1886^{(6)}$	$0.0571^{(3)}$	$0.0133^{(2)}$	$0.1137^{(5)}$	$0.0737^{(4)}$
	MSE	φ	$0.1673^{(3)}$	$0.1371^{(1)}$	$0.1629^{(2)}$	$0.2318^{(6)}$	$0.1944^{(5)}$	$0.1694^{(4)}$
		η	$0.2130^{(1)}$	$0.3034^{(6)}$	$0.2384^{(2)}$	$0.2459^{(3)}$	$0.2920^{(5)}$	$0.2517^{(4)}$
	$\sum ranks$		$\mathbf{16^{(1)}}$	$\mathbf{21^{(3)}}$	$\mathbf{18^{(2)}}$	$\mathbf{25^{(6)}}$	$\mathbf{24^{(5)}}$	$\mathbf{22^{(4)}}$
100	AVE	φ	$3.0319^{(5)}$	$2.9739^{(1)}$	$3.0218^{(2.5)}$	$3.0408^{(6)}$	$3.0128^{(2.5)}$	$3.0186^{(4)}$
		η	$1.5166^{(1)}$	$1.6120^{(6)}$	$1.5410^{(3)}$	$1.5236^{(2)}$	$1.5736^{(5)}$	$1.5458^{(4)}$
	AVAB	φ	$0.0319^{(5)}$	$0.0261^{(1)}$	$0.0218^{(2.5)}$	$0.0408^{(6)}$	$0.0128^{(2.5)}$	$0.0186^{(4)}$
		η	$0.0166^{(1)}$	$0.1120^{(6)}$	$0.0410^{(3)}$	$0.0236^{(2)}$	$0.0736^{(5)}$	$0.0458^{(4)}$
	MSE	φ	$0.0701^{(2)}$	$0.0638^{(1)}$	$0.0738^{(3)}$	$0.0909^{(6)}$	$0.0844^{(5)}$	$0.0756^{(4)}$
		η	$0.1062^{(1)}$	$0.1322^{(5)}$	$0.1172^{(2)}$	$0.1295^{(4)}$	$0.1423^{(6)}$	$0.1201^{(3)}$
	$\sum ranks$		$\mathbf{15^{(1)}}$	$\mathbf{20^{(3)}}$	$\mathbf{16^{(2)}}$	$\mathbf{26^{(5.5)}}$	$\mathbf{26^{(5.5)}}$	$\mathbf{23^{(4)}}$
300	AVE	φ	$3.0093^{(5)}$	$2.9869^{(1)}$	$3.0064^{(4)}$	$3.0129^{(6)}$	$3.0039^{(2)}$	$3.0061^{(3)}$
		η	$1.4983^{(1)}$	$1.5329^{(6)}$	$1.5057^{(4)}$	$1.4996^{(2)}$	$1.5158^{(5)}$	$1.5051^{(3)}$
	AVAB	φ	$0.0093^{(4)}$	$0.0131^{(6)}$	$0.0064^{(3)}$	$0.0129^{(5)}$	$0.0039^{(1)}$	$0.0061^{(2)}$
		η	$0.0017^{(2)}$	$0.0329^{(6)}$	$0.0057^{(4)}$	$0.0004^{(1)}$	$0.0158^{(5)}$	$0.0051^{(3)}$
	MSE	φ	$0.0223^{(2)}$	$0.0217^{(1)}$	$0.0233^{(3)}$	$0.0266^{(6)}$	$0.0260^{(5)}$	$0.0235^{(4)}$
		η	$0.0307^{(1)}$	$0.0330^{(4)}$	$0.0332^{(2.5)}$	$0.0385^{(5)}$	$0.0395^{(6)}$	$0.0335^{(2.5)}$
	$\sum ranks$		$\mathbf{15^{(1)}}$	$\mathbf{24^{(4.5)}}$	$\mathbf{20.5^{(3)}}$	$\mathbf{25^{(6)}}$	$\mathbf{24^{(4.5)}}$	$\mathbf{17.5^{(2)}}$
1000	AVE	φ	$3.0018^{(5)}$	$2.9930^{(1)}$	$3.0010^{(3.5)}$	$3.0026^{(6)}$	$2.9999^{(2)}$	$3.0010^{(3.5)}$
		η	$1.4943^{(1)}$	$1.5049^{(6)}$	$1.4966^{(4)}$	$1.4956^{(2)}$	$1.5005^{(5)}$	$1.4962^{(3)}$
	AVAB	φ	$0.0018^{(4)}$	$0.0070^{(6)}$	$0.0010^{(2.5)}$	$0.0026^{(5)}$	$0.00001^{(1)}$	$0.0010^{(2.5)}$
		η	$0.0057^{(6)}$	$0.0049^{(5)}$	$0.0034^{(2)}$	$0.0044^{(4)}$	$0.00005^{(1)}$	$0.0038^{(3)}$
	MSE	φ	$0.0063^{(2)}$	$0.0062^{(1)}$	$0.0067^{(3.5)}$	$0.0076^{(6)}$	$0.0075^{(5)}$	$0.0067^{(3.5)}$
		η	$0.0091^{(1.5)}$	$0.0091^{(1.5)}$	$0.0098^{(3.5)}$	$0.0114^{(5.5)}$	$0.0114^{(5.5)}$	$0.0098^{(3.5)}$
	$\sum ranks$		$\mathbf{19.5^{(3.5)}}$	$\mathbf{20.5^{(5)}}$	$\mathbf{19^{(1.5)}}$	$\mathbf{28.5^{(6)}}$	$\mathbf{19.5^{(3.5)}}$	$\mathbf{19^{(1.5)}}$

Table 6. Simulation results of the six estimation methods for $\varphi = 3, \eta = 2$.

n	Measures	Pa.	MLEs	MPSEs	ANDE	CVME	LSE	WLSE
50	AVE	φ	$3.0815^{(5)}$	$2.9773^{(1)}$	$3.0586^{(4)}$	$3.1025^{(6)}$	$3.0428^{(2)}$	$3.0500^{(3)}$
		η	$2.0110^{(1)}$	$2.2514^{(6)}$	$2.0761^{(3)}$	$2.0178^{(2)}$	$2.1516^{(5)}$	$2.0983^{(4)}$
	AVAB	φ	$0.0814^{(5)}$	$0.0227^{(1)}$	$0.0586^{(4)}$	$0.1025^{(6)}$	$0.0428^{(2)}$	$0.0500^{(3)}$
		η	$0.0110^{(1)}$	$0.2514^{(6)}$	$0.0761^{(3)}$	$0.0178^{(2)}$	$0.1516^{(5)}$	$0.0983^{(4)}$
	MSE	φ	$0.1675^{(3)}$	$0.1373^{(1)}$	$0.1629^{(2)}$	$0.2318^{(6)}$	$0.1944^{(5)}$	$0.1694^{(4)}$
		η	$0.3785^{(1)}$	$0.5400^{(6)}$	$0.4238^{(2)}$	$0.4372^{(3)}$	$0.5191^{(5)}$	$0.4474^{(4)}$
	$\sum ranks$		$\mathbf{16^{(1)}}$	$\mathbf{21^{(3)}}$	$\mathbf{18^{(2)}}$	$\mathbf{25^{(6)}}$	$\mathbf{24^{(5)}}$	$\mathbf{22^{(4)}}$
100	AVE	φ	$3.0319^{(5)}$	$2.9746^{(1)}$	$3.0218^{(4)}$	$3.0408^{(6)}$	$3.0128^{(2)}$	$3.0186^{(3)}$
		η	$2.0221^{(1)}$	$2.1494^{(6)}$	$2.0547^{(3)}$	$2.0314^{(2)}$	$2.0981^{(5)}$	$2.0611^{(4)}$
	AVAB	φ	$0.0319^{(5)}$	$0.0254^{(4)}$	$0.0218^{(2.5)}$	$0.0408^{(6)}$	$0.0128^{(2.5)}$	$0.0186^{(1)}$
		η	$0.0221^{(1)}$	$0.1494^{(6)}$	$0.0547^{(3)}$	$0.0314^{(2)}$	$0.0981^{(5)}$	$0.0611^{(4)}$
	MSE	φ	$0.0701^{(2)}$	$0.0637^{(1)}$	$0.0738^{(3)}$	$0.0909^{(6)}$	$0.0844^{(5)}$	$0.0756^{(4)}$
		η	$0.1887^{(1)}$	$0.2349^{(5)}$	$0.2084^{(2)}$	$0.2303^{(4)}$	$0.2530^{(6)}$	$0.2135^{(3)}$
	$\sum ranks$		$\mathbf{15^{(1)}}$	$\mathbf{23^{(4)}}$	$\mathbf{17.5^{(2)}}$	$\mathbf{26^{(6)}}$	$\mathbf{25.5^{(5)}}$	$\mathbf{19^{(3)}}$
300	AVE	φ	$3.0093^{(5)}$	$2.9858^{(1)}$	$3.0064^{(4)}$	$3.0129^{(6)}$	$3.0039^{(2)}$	$3.0061^{(3)}$
		η	$1.9977^{(1)}$	$2.0435^{(6)}$	$2.0075^{(4)}$	$1.9994^{(2)}$	$2.0210^{(5)}$	$2.0069^{(3)}$
	AVAB	φ	$0.0093^{(4)}$	$0.0142^{(6)}$	$0.0064^{(3)}$	$0.0129^{(5)}$	$0.0039^{(1)}$	$0.0061^{(2)}$
		η	$0.0023^{(2)}$	$0.0435^{(6)}$	$0.0075^{(4)}$	$0.0006^{(1)}$	$0.0210^{(5)}$	$0.0069^{(3)}$
	MSE	φ	$0.0224^{(2)}$	$0.0216^{(1)}$	$0.0233^{(3)}$	$0.0266^{(6)}$	$0.0260^{(5)}$	$0.0235^{(4)}$
		η	$0.0545^{(1)}$	$0.0591^{(2.5)}$	$0.0591^{(2.5)}$	$0.0684^{(5)}$	$0.0702^{(6)}$	$0.0595^{(4)}$
	$\sum ranks$		$\mathbf{15^{(1)}}$	$\mathbf{22.5^{(4)}}$	$\mathbf{20.5^{(3)}}$	$\mathbf{25^{(6)}}$	$\mathbf{24^{(5)}}$	$\mathbf{19^{(2)}}$
1000	AVE	φ	$3.0018^{(5)}$	$2.9931^{(1)}$	$3.0010^{(3.5)}$	$3.0026^{(6)}$	$2.9999^{(2)}$	$3.0010^{(3.5)}$
		η	$1.9923^{(1)}$	$2.0071^{(6)}$	$1.9955^{(4)}$	$1.9942^{(2)}$	$2.0006^{(5)}$	$1.9949^{(3)}$
	AVAB	φ	$0.0018^{(4)}$	$0.0069^{(6)}$	$0.0010^{(2.5)}$	$0.0026^{(5)}$	$0.00001^{(1)}$	$0.0010^{(2.5)}$
		η	$0.0077^{(6)}$	$0.0071^{(5)}$	$0.0045^{(2)}$	$0.0058^{(4)}$	$0.00006^{(1)}$	$0.0051^{(3)}$
	MSE	φ	$0.0063^{(2)}$	$0.0061^{(1)}$	$0.0067^{(3.5)}$	$0.0076^{(6)}$	$0.0075^{(5)}$	$0.0067^{(3.5)}$
		η	$0.0161^{(1)}$	$0.0164^{(2)}$	$0.0174^{(3.5)}$	$0.0203^{(5.5)}$	$0.0203^{(5.5)}$	$0.0174^{(3.5)}$
	$\sum ranks$		$\mathbf{20^{(4)}}$	$\mathbf{21^{(5)}}$	$\mathbf{19^{(1.5)}}$	$\mathbf{28.5^{(6)}}$	$\mathbf{19.5^{(3)}}$	$\mathbf{19^{(1.5)}}$

The results in Table 1 - 6 shows that the mean square errors (MSEs) decrease for all parameter combinations as the sample size increases, that is, all the estimators are quite consistent. Hence, the estimates of the $TIHL_{ST}$ parameters provide credible MSEs and low AVABs for all the parameter combinations using the six estimation methods. Additionally, the average absolute biases (AVABs) approach zero as the sample size increase, proving that the estimators are quite asymptotically unbiased.

Table 7. Partial and overall ranks of the six methods of estimation for all parameter combinations.

Parameter	n	MLE	MPSE	ANDE	CVME	LSE	WLSE
$\varphi = 2, \eta = 0.5$	50	1	3	2	4.5	6	4.5
	100	1	4	2.5	5	6	2.5
	300	1	4	2.5	5.5	5.5	2.5
	1000	2	1	4	5.5	5.5	3
$\varphi = 2, \eta = 1.5$	50	1	3	2	4.5	6	4.5
	100	1	4	2.5	5	6	2.5
	300	1	4	3	5.5	5.5	2
	1000	2	1	3.5	5.5	5.5	3.5
$\varphi = 2, \eta = 2$	50	1	3	2	4.5	6	4.5
	100	1	4	2.5	5	6	2.5
	300	1	4	2.5	5.5	5.5	2.5
	1000	1	2	3.5	6	5	3.5
$\varphi = 3, \eta = 0.5$	50	1	3	2	6	5	4
	100	1.5	3	1.5	6	5	4
	300	1	4.5	3	6	4.5	2
	1000	3	4.5	1.5	6	4.5	1.5
$\varphi = 3, \eta = 1.5$	50	1	3	2	6	5	4
	100	1	3	2	5.5	5.5	4
	300	1	4.5	3	6	4.5	2
	1000	3.5	5	1.5	6	3.5	1.5
$\varphi = 3, \eta = 2$	50	1	3	2	6	5	4
	100	1	4	2	6	5	3
	300	1	4	3	6	5	2
	1000	4	5	1.5	6	3	1.5
$\sum ranks$		**34**	**83.5**	**57.5**	**133.5**	**124**	**71.5**
Overall rank		**1**	**4**	**2**	**6**	**5**	**3**

The performance ordering of the six estimators from best to worst, using the overall ranks in Table 7 is MLE, ANDE, WLSE, MPSE, LSE, and CVME for all the parameter combinations. In summary, the simulation results show that the six estimators in estimating the $TIHL_{ST}$ parameters perform very well. We can conclude that MLE outperform all the other estimators with an overall rank of 34. Therefore, based on our work, we can confirm the superiority of MLE and ANDE, with overall ranks of 34 and 57.5 for the $TIHL_{ST}$ distribution.

IV. Conclusion

In this work, we considered the estimation of the type I half logistic skew-t ($TIHL_{ST}$) model parameters using six classical estimation procedures, namely the maximum likelihood, least squares, weighted least squares, Anderson-Darling, maximum product of spacing and Cramer-von mises. Given that the theoretical comparison between these classical estimation procedures is not quite practicable, an extensive Monte Carlo simulation study is performed to compare the criterions in terms of average absolute biases, mean square error of each estimate and average parameter estimates. The results show that the criterions performance ordering from finest to poorest, using the overall ranks is MLE, ANDE, WLSE, MPSE, LSE, and CVME for all the parameter combinations. We can also state that the MLE outperform all the other criterions with an overall rank of 34. Consequently, we confirm the dominance of the MLE with overall rank of 34 and ANDE with overall rank of 57.5, for the $TIHL_{ST}$ distribution.

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