Bayesian and Maximum Likelihood Estimations of the Scale Parameter of a New Weighted Weibull Distribution

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***Abstract* — The new weighted Weibull distribution is a two-parameter lifetime model with high flexibility for analyzing real-life data. It has a scale parameter and a shape parameter responsible for the flexibility in the distribution. With all the importance and necessity of parameter estimation theory in model fitting and application, it has not been established that a particular estimation method is better for any of these two parameters of the new weighted Weibull distribution. Therefore, this paper focuses on the development of Bayesian estimators for the scale parameter of the new weighted Weibull distribution using a non-informative prior distribution (Jeffery) under the Quadratic loss function (QLF). These results are compared with the maximum likelihood estimation method using Monte Carlo simulations. To compare the efficiency of the two estimation methods, the mean square error (MSE) has been used as a criterion for choosing the best estimator.**

Keywords - *Weighted Weibull distribution; Bayesian analysis; Jeffrey Prior; Quadratic Loss function; MLE; Mean square error.*

I. Introduction

Some standard probability distributions have been used over the years for modeling real-life datasets however research has shown that most of these distributions do not adequately model some of these heavily skewed datasets and therefore creating a problem in statistical theory and applications. Recently, numerous extended or compound probability distributions have been proposed in the literature for modeling real-life situations and these compound distributions are found to be skewed, flexible and better in statistical modeling compared to their standard counterparts ([1]- [14]).

Due to the fact above, [15] developed a new weighted Weibull distribution (NWWD) with three parameters (two shapes parameters and a scale parameter). This distribution is skewed and flexible with an increasing hazard rate and different shapes and also performed better than the Weibull distribution based on applications of the models to three-lifetime datasets [15].

In [15], the probability density function (pdf), the cumulative distribution function (cdf), survival function, hazard function, and quantile function (qf) of the NWWD are respectively defined as:

$$f(x) = \left(1+\beta^{\theta}\right)\alpha\theta x^{\theta-1}\mathrm{e}^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)} \tag{1.1}$$

$$F(x) = 1-\mathrm{e}^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)} \tag{1.2}$$

$$S(x) = \mathrm{e}^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)} \tag{1.3}$$

$$h(x) = \left(1+\beta^{\theta}\right)\alpha\theta x^{\theta-1} \tag{1.4}$$

and

$$Q(u) = \left(-\frac{\ln(1-u)}{\alpha\left(1+\beta^{\theta}\right)}\right)^{\frac{1}{\theta}} \tag{1.5}$$

A graphical representation of the above functions using some arbitrary parameter values is displayed in Figure 1:

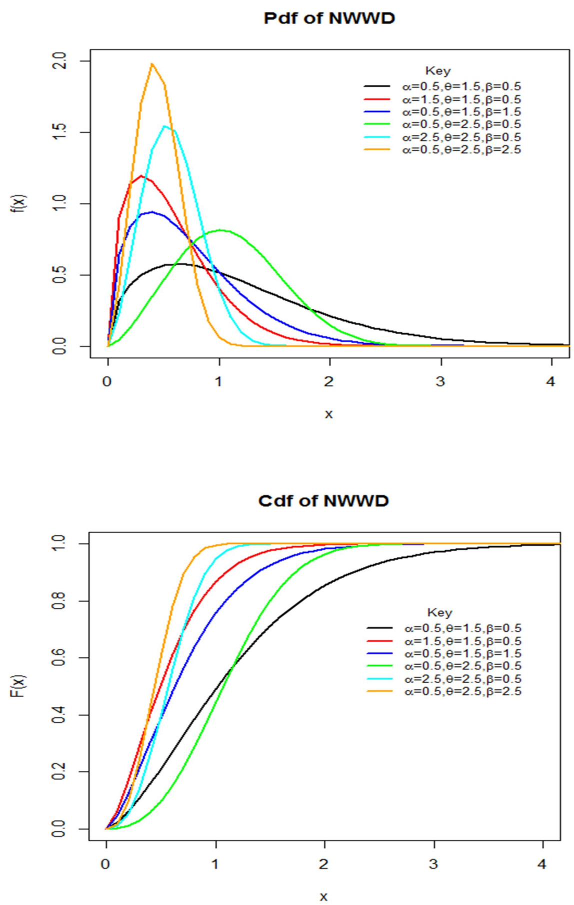

Fig. 1: Plots of the PDF and of the NWWD for Selected Parameter Values.

Estimation of parameters in a distribution differs by method from one parameter of the distribution to another and therefore this study aims at estimating the scale parameter of the NWWD using Bayesian approach and making a comparison between the Bayesian approach and the method of maximum likelihood estimation.

The aim of this article is to estimate the scale parameter of the NWWD using Bayesian approach assuming a Jeffrey's prior distribution with a quadratic loss function. Next to this introductory section is the remaining part of this paper presented as follows: in Section 2, maximum likelihood estimator (MLE) for the scale parameter is obtained. In Section 3, Bayesian estimators based on the quadratic loss function by assuming a Jeffrey's prior distribution is derived. The proposed estimators are compared using their mean squared error (MSE) in Section 4. Finally, the conclusion is made in Section 5.

II. Research Methodology

1. Maximum Likelihood Estimation

Let $X_1, X_2, ..., X_n$ be a random sample from a population X of size 'n' independently and identically distributed random variables with probability density function $f(x)$, . The likelihood is the joint probability function of the data, but viewed as a function of the parameters, treating the observed data as fixed quantities. Given that the values, $\underline{x} = (x_1, x_2, ..., x_n)$ are obtained independently from the NWWD with unknown parameters, α , θ and β .

The likelihood function is given by:

$$L(\underline{x}\,|\,\alpha,\theta,\beta) = P(x_1,x_2,...,x_n\,|\,\alpha,\theta,\beta) = \prod_{i=1}^{n} P(\underline{x}\,|\,\alpha,\theta,\beta) \quad (2.1)$$

The likelihood function, $L(\underline{x}\,|\,\alpha,\theta,\beta)$ based on the pdf of NWWD is defined to be the joint density of the random variables $x_1, x_2, ..., x_n$ and it is given as:

$$L(\underline{x}\,|\,\alpha,\theta,\beta) = \left(1+\beta^{\theta}\right)^{n} (\alpha\theta)^{n} \prod_{i=1}^{n} x_i^{\theta-1} \mathrm{e}^{-\alpha\left(1+\beta^{\theta}\right)\sum_{i=1}^{n} x_i^{\theta}} \quad (2.2)$$

For the scale parameter of the NWWD, α , the likelihood function is given by;

$$L(\underline{x}\,|\,\alpha) \propto (\alpha)^{n}\, \mathrm{e}^{-\alpha\left(1+\beta^{\theta}\right)\sum_{i=1}^{n} x_i^{\theta}}$$

$$L(\underline{x}\,|\,\alpha) = \eta(\alpha)^{n}\, \mathrm{e}^{-\alpha\left(1+\beta^{\theta}\right)\sum_{i=1}^{n} x_i^{\theta}} \quad (2.3)$$

where $\eta = \left(1+\beta^{\theta}\right)^{n} (\theta)^{n} \prod_{i=1}^{n} x_i^{\theta-1}$ is a constant which is independent of the scale parameter, α .

Let the log-likelihood function, $l = \log L(\underline{x}\,|\,\alpha)$, therefore

$$\log L(\underline{x}\,|\,\alpha) = n\log\alpha - \alpha\left(1+\beta^{\theta}\right)\sum_{i=1}^{n} x_i^{\theta} \quad (2.4)$$

Differentiating ***l*** partially with respect to α gives;

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - \left(1+\beta^{\theta}\right)\sum_{i=1}^{n} x_i^{\theta} = 0 \qquad (2.5)$$

And solving for $\hat{\alpha}$ gives;

$$\hat{\alpha} = n\left(\left(1+\beta^{\theta}\right)\sum_{i=1}^{n} x_i^{\theta}\right)^{-1} \qquad (2.6)$$

where $\hat{\alpha}$ is the maximum likelihood estimator of the scale parameter, α. Details concerning the maximum likelihood estimators of the scale parameter of the NWWD can be found in [15].

2. Bayesian Estimation

The Bayesian inference requires an appropriate choice of prior(s) for the parameter(s). From the Bayesian viewpoint, there is no clear-cut way from which one can conclude that one prior is better than the other. Nevertheless, very often priors are chosen according to one's subjective knowledge and beliefs. However, if one has adequate information about the parameter(s), it is better to choose informative prior(s); otherwise, it is preferable to use non-informative prior(s).

In this study, a non-informative prior (Jeffrey) will be considered for estimating the scale parameter of the NWWD. This assumed prior distribution has been used widely by several authors including [19]-[27]. This study also considers the quadratic loss function which has also been used previously by some researchers such as [28]-[38] etc.

The posterior distribution of a parameter is the distribution of the parameter after observing the available data and it is obtained by using Bayes' theorem in relation to the scale parameter α, likelihood function and prior distribution as follows:

$$P(\alpha|\underline{x}) = \frac{P(\alpha,\underline{x})}{P(\underline{x})} = \frac{P(\underline{x}|\alpha)P(\alpha)}{P(\underline{x})} = \frac{P(\underline{x}|\alpha)P(\alpha)}{\int P(\underline{x}|\alpha)P(\alpha)d\alpha} = \frac{L(\underline{x}|\alpha)P(\alpha)}{\int L(\underline{x}|\alpha)P(\alpha)d\alpha}$$

(3.1)

where $P(\underline{x})$ is the marginal distribution of X and $P(\underline{x}) = \sum_{x}^{\infty} p(\alpha)L(\underline{x}|\alpha)$ when the prior distribution of α is discrete and $P(\underline{x}) = \int_{-\infty}^{\infty} p(\alpha)L(\underline{x}|\alpha)d\alpha$ when the prior distribution of α is continuous. Also, note that $p(\alpha)$ and $L(\underline{x}|\alpha)$ are the prior distribution and the Likelihood function respectively.

2.1 *Bayesian Estimation under Jeffrey's Prior with Quadratic Loss Functions*

The posterior distribution of the scale parameter α for a given data assuming Jeffrey's prior distribution is obtained from (3.1) using integration by substitution method as follows:

The Jeffrey's prior as a non-informative prior relating to the scale parameter α of the NWWD is defined as:

$$p(\alpha) \propto \frac{1}{\alpha}; 0 < \alpha < \infty$$

(3.2)

The posterior distribution of the scale parameter α for a given data using Jeffrey's prior is defined as:

$$P(\alpha|\underline{x}) = \frac{P(\alpha,\underline{x})}{P(\underline{x})} = \frac{P(\underline{x}|\alpha)P(\alpha)}{P(\underline{x})} = \frac{P(\underline{x}|\alpha)P(\alpha)}{\int P(\underline{x}|\alpha)P(\alpha)d\alpha} = \frac{L(\underline{x}|\alpha)P(\alpha)}{\int L(\underline{x}|\alpha)P(\alpha)d\alpha}$$

(3.3)

Now, let

$$K = \int L(\underline{x}|\alpha)P(\alpha)d\alpha \qquad (3.4)$$

Substituting for $P(\alpha)$ and $L(\underline{x}|\alpha)$; we have:

$$K = \eta\int_{0}^{\infty} \alpha^{n-1} e^{-\alpha\left(1+\beta^{\theta}\right)\sum_{i=1}^{n} x_i^{\theta}} d\alpha \qquad (3.5)$$

Also, using integration by substitution method in equation (3.5); we obtain the following:

Let

$$u = \alpha\left(1+\beta^{\theta}\right)\sum_{i=1}^{n} x_i^{\theta} \Rightarrow \alpha = \frac{u}{\left(1+\beta^{\theta}\right)\sum_{i=1}^{n} x_i^{\theta}}$$

$$\frac{du}{d\alpha} = \left(1+\beta^{\theta}\right)\sum_{i=1}^{n} x_i^{\theta} \Rightarrow d\alpha = \frac{du}{\left(1+\beta^{\theta}\right)\sum_{i=1}^{n} x_i^{\theta}}$$

Substituting for α and $d\alpha$ in equation (3.5) and simplifying; we have:

$$K=\eta\int_0^\infty\left(\frac{u}{\left(1+\beta^\theta\right)\sum_{i=1}^n x_i^\theta}\right)^{n-1}e^{-u}\frac{du}{\left(1+\beta^\theta\right)\sum_{i=1}^n x_i^\theta}$$

$$K=\eta\frac{1}{\left[\left(1+\beta^\theta\right)\sum_{i=1}^n x_i^\theta\right]^n}\int_0^\infty u^{n-1}e^{-u}du$$

(3.6)

Also recall that $\int_0^\infty t^{k-1}e^{-t}dt=\Gamma(k)$ and that

$$\int_0^\infty t^k e^{-t}dt=\int_0^\infty t^{k+1-1}e^{-t}dt=\Gamma(k+1)$$

Hence;

$$K=\frac{\eta\Gamma(n)}{\left[\left(1+\beta^\theta\right)\sum_{i=1}^n x_i^\theta\right]^n} \qquad (3.7)$$

Substituting for K, $P(\alpha)$ and $L(\underset{\sim}{x}\mid\alpha)$ in equation (3.3) and simplifying; we obtain the posterior distribution under Jeffrey's prior as follows:

$$P(\alpha\mid\underset{\sim}{x})=\frac{\eta\alpha^{n-1}e^{-\alpha\left(1+\beta^\theta\right)\sum_{i=1}^n x_i^\theta}}{\dfrac{\eta\Gamma(n)}{\left[\left(1+\beta^\theta\right)\sum_{i=1}^n x_i^\theta\right]^n}}$$

$$P(\alpha\mid\underset{\sim}{x})=\frac{\left[\left(1+\beta^\theta\right)\sum_{i=1}^n x_i^\theta\right]^n\alpha^{n-1}e^{-\alpha\left(1+\beta^\theta\right)\sum_{i=1}^n x_i^\theta}}{\Gamma(n)}$$

(3.8)

The derivation of Bayes estimator using *QLF* under Jeffrey's prior is obtained as:

$$\alpha_{QLF}=\frac{E\left(\alpha^{-1}\mid\underset{\sim}{x}\right)}{E\left(\alpha^{-2}\mid\underset{\sim}{x}\right)}=\frac{\int_0^\infty\alpha^{-1}P(\alpha\mid\underset{\sim}{x})d\alpha}{\int_0^\infty\alpha^{-2}P(\alpha\mid\underset{\sim}{x})d\alpha}$$

$$E\left(\alpha^{-1}\mid\underset{\sim}{x}\right)=\int_0^\infty\alpha^{-1}P(\alpha\mid\underset{\sim}{x})d\alpha \qquad (3.9)$$

Now recall that for Jeffrey's prior,

$$P(\alpha\mid\underset{\sim}{x})=\frac{\left[\left(1+\beta^\theta\right)\sum_{i=1}^n x_i^\theta\right]^n\alpha^{n-1}e^{-\alpha\left(1+\beta^\theta\right)\sum_{i=1}^n x_i^\theta}}{\Gamma(n)}$$

Substituting for $P(\alpha\mid\underset{\sim}{x})$ in equation (3.9); we have:

$$E\left(\alpha^{-1}\mid\underset{\sim}{x}\right)=\int_0^\infty\alpha^{-1}\frac{\left[\left(1+\beta^\theta\right)\sum_{i=1}^n x_i^\theta\right]^n\alpha^{n-1}e^{-\alpha\left(1+\beta^\theta\right)\sum_{i=1}^n x_i^\theta}}{\Gamma(n)}d\alpha$$

$$E\left(\alpha^{-1}\mid\underset{\sim}{x}\right)=\frac{\left[\left(1+\beta^\theta\right)\sum_{i=1}^n x_i^\theta\right]^n}{\Gamma(n)}\int_0^\infty\alpha^{n-2}e^{-\alpha\left(1+\beta^\theta\right)\sum_{i=1}^n x_i^\theta}d\alpha$$

(3.10)

Using integration by substitution in equation (3.10); we obtain the following:

Let

$$u=\alpha\left(1+\beta^\theta\right)\sum_{i=1}^n x_i^\theta\Rightarrow\alpha=\frac{u}{\left(1+\beta^\theta\right)\sum_{i=1}^n x_i^\theta}$$

$$\frac{du}{d\alpha}=\left(1+\beta^\theta\right)\sum_{i=1}^n x_i^\theta\Rightarrow d\alpha=\frac{du}{\left(1+\beta^\theta\right)\sum_{i=1}^n x_i^\theta}$$

Substituting for α and $d\alpha$ in equation (3.10) and simplifying; we have:

$$E\left(\alpha^{-1}\mid\underset{\sim}{x}\right)=\frac{\left[\left(1+\beta^\theta\right)\sum_{i=1}^n x_i^\theta\right]}{(n-1)}$$

(3.11)

Similarly;

$$E\left(\alpha^{-2}\mid\underset{\sim}{x}\right)=\int_0^\infty\alpha^{-2}P(\alpha\mid\underset{\sim}{x})d\alpha$$

(3.12)

Now recall that for Jeffrey's prior,

$$P(\alpha \mid \underline{x}) = \frac{\left[\left(1+\beta^{\theta}\right)\sum_{i=1}^{n} x_i^{\theta}\right]^n \alpha^{n-1} e^{-\alpha\left(1+\beta^{\theta}\right)\sum_{i=1}^{n} x_i^{\theta}}}{\Gamma(n)}$$

Substituting for $P(\alpha \mid \underline{x})$ in equation (3.12); we have:

$$E\left(\alpha^{-2} \mid \underline{x}\right) = \int_0^{\infty} \alpha^{-2} \frac{\left[\left(1+\beta^{\theta}\right)\sum_{i=1}^{n} x_i^{\theta}\right]^n \alpha^{n-1} e^{-\alpha\left(1+\beta^{\theta}\right)\sum_{i=1}^{n} x_i^{\theta}}}{\Gamma(n)} d\alpha$$

$$E\left(\alpha^{-2} \mid \underline{x}\right) = \frac{\left[\left(1+\beta^{\theta}\right)\sum_{i=1}^{n} x_i^{\theta}\right]^n}{\Gamma(n)} \int_0^{\infty} \alpha^{n-3} e^{-\alpha\left(1+\beta^{\theta}\right)\sum_{i=1}^{n} x_i^{\theta}} d\alpha$$

(3.13)

Using integration by substitution in equation (3.13); we obtain the following:

Let

$$u = \alpha\left(1+\beta^{\theta}\right)\sum_{i=1}^{n} x_i^{\theta} \Rightarrow \alpha = \frac{u}{\left(1+\beta^{\theta}\right)\sum_{i=1}^{n} x_i^{\theta}}$$

$$\frac{du}{d\alpha} = \left(1+\beta^{\theta}\right)\sum_{i=1}^{n} x_i^{\theta} \Rightarrow d\alpha = \frac{du}{\left(1+\beta^{\theta}\right)\sum_{i=1}^{n} x_i^{\theta}}$$

Substituting for α and $d\alpha$ in equation (3.13) and simplifying; we have:

$$E\left(\alpha^{-2} \mid \underline{x}\right) = \frac{\left[\left(1+\beta^{\theta}\right)\sum_{i=1}^{n} x_i^{\theta}\right]^2}{(n-1)(n-2)}$$

(3.14)

But recall that

$$\alpha_{QLF} = \frac{E\left(\alpha^{-1} \mid \underline{x}\right)}{E\left(\alpha^{-2} \mid \underline{x}\right)}$$

This implies that

$$\alpha_{QLF} = \frac{\left[\left(1+\beta^{\theta}\right)\sum_{i=1}^{n} x_i^{\theta}\right]}{(n-1)} \div \frac{\left[\left(1+\beta^{\theta}\right)\sum_{i=1}^{n} x_i^{\theta}\right]^2}{(n-1)(n-2)}$$

$$\alpha_{QLF} = \frac{(n-2)}{\left(1+\beta^{\theta}\right)\sum_{i=1}^{n} x_i^{\theta}}$$

(3.15)

III. Results And Discussion

A. Results

In this section, Monte Carlo simulation with R software under 10,000 replications is considered to generate random samples of sizes n = (25, 50, 75, 100, 125, 150) from the NWWD using the quantile function (inverse transformation method of simulation) under the following combination of parameter values: $\alpha = 0.5, \theta = 0.5, \beta = 0.5$, $\alpha = 0.5, \theta = 1.5, \beta = 0.5$, $\alpha = 1.5, \theta = 0.5, \beta = 0.5$ and $\alpha = 0.5, \theta = 0.5, \beta = 1.5$. The following table presents the results of a simulation study by listing the average estimates of the scale parameter with their respective Mean Square Errors (MSEs) under the appropriate estimation methods which include the Maximum Likelihood Estimation (*MLE*) and Quadratic Loss Function (*QLF*) under Jeffrey prior respectively. The criterion for evaluating the performance of the estimators in this study is the Mean Square Error (MSE): $MSE = \frac{1}{n} E\left(\hat{\alpha} - \alpha\right)^2$.

Table 4.1: Estimates and Mean Squared Errors (within parenthesis) for $\hat{\alpha}$ under Jeffrey's prior. MLE=Maximum likelihood estimator, QLF= Quadratic loss function.

Size (n)	Parameter (True values)			Methods of Estimation	
	α	θ	β	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{QLF}$
25	0.5	0.5	0.5	0.5204 (0.0123)	0.4788 (0.0105)
	0.5	1.5	0.5	1.5613 (0.1106)	1.4364 (0.0945)
	1.5	0.5	0.5	1.5613 (0.1106)	1.4364 (0.0945)
	0.5	0.5	1.5	0.5204 (0.0123)	0.4788 (0.0105)
50	0.5	0.5	0.5	0.5095 (0.0054)	0.4891 (0.0050)
	0.5	1.5	0.5	1.5286 (0.0489)	1.4674 (0.0454)
	1.5	0.5	0.5	1.5286 (0.0489)	1.4674 (0.0454)
	0.5	0.5	1.5	0.5095 (0.0054)	0.4891 (0.0050)
75	0.5	0.5	0.5	0.5073 (0.0036)	0.4938 (0.0034)
	0.5	1.5	0.5	1.5219 (0.0326)	1.4813 (0.0308)
	1.5	0.5	0.5	1.5219 (0.0326)	1.4813 (0.0308)
	0.5	0.5	1.5	0.5073 (0.0036)	0.4938 (0.0034)
100	0.5	0.5	0.5	0.5053 (0.0027)	0.4952 (0.0026)
	0.5	1.5	0.5	1.5158 (0.0240)	1.4855 (0.0230)
	1.5	0.5	0.5	1.5158 (0.0240)	1.4855 (0.0230)
	0.5	0.5	1.5	0.5053 (0.0027)	0.4952 (0.0026)
125	0.5	0.5	0.5	0.5046 (0.0021)	0.4965 (0.0020)
	0.5	1.5	0.5	1.5137 (0.0189)	1.4895 (0.0183)
	1.5	0.5	0.5	1.5137 (0.0189)	1.4895 (0.0183)
	0.5	0.5	1.5	0.5046 (0.0021)	0.4965 (0.0020)
150	0.5	0.5	0.5	0.5035 (0.0017)	0.4968 (0.0017)
	0.5	1.5	0.5	1.5104 (0.0153)	1.4903 (0.0149)
	1.5	0.5	0.5	1.5104 (0.0153)	1.4903 (0.0149)
	0.5	0.5	1.5	0.5035 (0.0017)	0.4968 (0.0017)

Looking at the results from Table 4.1, one can see that the estimators of the scale parameter using QLF under Jeffrey prior is better than the MLEs based on the fact that it has the lowest MSE despite the changes in the samples and chosen parameter values. This consistency in the result for Bayesian estimators (using QLF under Jeffrey prior) is proof that the approach is more efficient for estimating the scale parameter compared to MLE.

Generally, the results in Table 4.1 have proven that the average estimates of the scale parameter get closer to the true parameter value when sample size increases, and the mean square errors (MSEs) all decrease as sample size increases which satisfies the first-order asymptotic theory. Similarly, Bayesian estimators and maximum likelihood estimators (MLEs) all become better when the sample size increases. In fact, for very large sample sizes the performances of these estimators are observed to be relatively the same for both methods of estimation.

IV. Conclusion

This paper has derived Bayesian estimators for the scale parameter of NWWD by assuming a Jeffrey prior distribution with the Quadratic Loss Function. Posterior distribution and Bayes estimators of this parameter are derived using the above prior and loss function. The efficiency of these estimators has been evaluated by means of their mean square errors using the inverse transformation method of Monte Carlo Simulations with different parameter values and sample sizes.

The results of the simulation and comparison show that using the quadratic loss function gives estimators with the lowest MSEs. Precisely, it is found that the Bayesian Method using Quadratic Loss Function under Jeffrey prior produces the best estimator of the scale parameter compared to the estimator of the Maximum Likelihood method irrespective of the chosen parameters values and the allocated sample sizes.

This research also found that the variation in the values of the shape parameters of the distribution does not affect or change the performance of the estimators of the estimated scale parameter, however, it is recommended that since this study considers only the scale parameter of the distribution, subsequent studies should consider any of the shape parameters of the distribution due to the fact that in statistical applications of this model it will be very important to identify and understand the best method for estimating both the scale and shape parameters of the model.

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