Trapezoid Fuzzy-Shewhart Control Chart Based on α-Level Mid-Range Transformation and its Senstivity Measures

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***Abstract* — The Shewhart $\bar{\bar{X}} - \bar{S}$ and $\bar{\bar{X}} - \bar{R}$ control chart are well known for their sensitivity to a large shift in the process mean and their insensitivity to small and moderate shift. This paper intended to determine the sensitivity of fuzzy Shewhart $\tilde{\bar{\bar{X}}} - \tilde{\bar{S}}$ and $\tilde{\bar{\bar{X}}} - \tilde{\bar{R}}$ control charts to small, moderate and large shift in industrial process. The results shows that the fuzzy Shewhart $\tilde{\bar{\bar{X}}} - \tilde{\bar{S}}$ and $\tilde{\bar{\bar{X}}} - \tilde{\bar{R}}$ control charts exhibit similar insensitivity towards the small and moderate shift in the process mean but very sensitive to large shift..**

Keywords- *Shewhart $\bar{\bar{X}} - \bar{S}$ and $\bar{\bar{X}} - \bar{R}$ control charts, Average run length, fuzzy Shewhart $\tilde{\bar{\bar{X}}} - \tilde{\bar{S}}$ and $\tilde{\bar{\bar{X}}} - \tilde{\bar{R}}$ control charts, fuzziness, defuzzification.*

I. Introduction

Statistical process control is an approach adopted to monitor the mean and the variability of any industrial production process. Control chart is one of the most important statistical process control tools used to diagnose, monitor and stabilize a process or product. As established by the Walter Shewhart (1920), that variations exist in quality in every production line. Among the cause of these two types of variation, one is seems to be random which can be considered to belong to the category of chance or common cause of variations which little or nothing may be done about it, other than to review the process. The other cause of variation in a process is assignable cause of variation, this arises as a result of special cause which can be traced, identified, removed the special cause and revert back to in-control process.

The Shewhart procedure involved precise measurement such as diameter of piston. But in some situation process characteristics measurement may imprecise, such measure could be very good piston, good, fair piston and bad piston. This form of measure is addressed as fuzzy variable as proposed by Zadeh (1965) in his fuzzy set theory. More often, some industrial process observation comes with such vague measurement.

Many control charts has been developed from the concept of the Shewhart idea and as such many of them has been compared vis-à-vis with the Shewhart chart. Recently, as the concept of the fuzzy control chart is gaining attention, more works are required to study it sensitivity to false alarm.

II. Literature Review

Various approaches have been reported regarding the concept of Shewhart control chart along with the zadeh idea especially in quality control studies. Such includes but not limited to Kanagawa *et al* (1993), Wang and Raz(1990), Rowlands and Wang (2000), El-shal and Moris (2000), Karnik and Mendel (2001), Gulbay *et al* (2004), Cheng (2005),Gulbay and Kahraman (2006a), Mendel (2007), Erginel (2008), Zarandi *et al* (2008), Senturk and Erginel (2009), Senturk *et al* (2010), Kaya and Kahraman (2011), Erginel *et al* (2011), Pandurangan (2011), Senturk *et al*

(2014) Erginel (2014), Poongodi Senturk (2010), and Muthulakshmi (2015), Wang and Hyrniewicz (2015), Edmundas *et al* (2015), Chen and Huang (2016), Hou *et al* (2016), Zafirah *et al* (2016), Kaya *et al* (2017), Sakthivel *et al* (2017), Akeem (2018). More work is still in progress on sensitivity of fuzzy-Shewhart control.

III. METHODOLOGY

Fuzzy Transformation Approach (α-Level Midrange):

Fuzzy number unlike the crisp number required transformation to make its results look crisp for better presentation. In this case, α-level fuzzy fuzzy transformation will be adopted even though it is one of the most used to defuzzifying the fuzzy number. Since the focus of this research does not rely on this transformation method but rather on the sensitivity of the Shewhart $\bar{\bar{X}} - \bar{S}$ and $\bar{\bar{X}} - \bar{R}$ control chart.

This α-level fuzzy midrange transformation approach used to build the fuzzy $\bar{\bar{X}} - \bar{R}$ control based on fuzzy trapezoid numbers. The method was developed by Senturk and Erginel (2009) using triangular fuzzy number. The α-Level Midrange is described as the midpoint of the end of α-level cuts, denoted by A^{α} is a non fuzzy set that comprises all elements whose membership is greater than or equal to α. If p^{α} and r^{α} are the end points of A^{α}, then

$$f_{mr}^{\alpha} = \frac{1}{2}(p^{\alpha} + r^{\alpha}) \qquad (1)$$

$s_{mr.j}^{\alpha}$ is used to transform the fuzzy control limits into scalar at α-level fuzzy midrange of sample j. and it is given as

$$s_{mr.j}^{\alpha} = \frac{(p_i+s_i)+\alpha[(q_i-p)-(s_i-r)]}{2} \qquad (2)$$

Fuzzy $\bar{\bar{X}} - \bar{R}$ Control Chart

From the traditional control limits for $\bar{\bar{X}} - \bar{R}$ Control Chart

$$UCL_{\bar{X}} = \bar{\bar{X}} + A_2\bar{R}$$
$$CL_{\bar{X}} = \bar{\bar{X}} \qquad (3)$$
$$LCL_{\bar{X}} = \bar{\bar{X}} - A_2\bar{R}$$

where A_2 is a control chart coefficient and $\bar{R}$ is the mean of the sample ranges R_j.

The control limits for fuzzy $\tilde{\bar{\bar{X}}} - \tilde{R}$ control chart based on fuzzy trapezoidal number are

$$\widetilde{UCL}_{\bar{X}} = \widetilde{CL} + A_2\bar{R} = (\bar{\bar{X}}_p, \bar{\bar{X}}_q, \bar{\bar{X}}_r, \bar{\bar{X}}_s) + A_2(\bar{R}_p, \bar{R}_q, \bar{R}_r, \bar{R}_s)$$
$$= (\bar{\bar{X}}_p + A_2\bar{R}_p, \bar{\bar{X}}_q + A_2\bar{R}_q, \bar{\bar{X}}_r + A_2\bar{R}_r, \bar{\bar{X}}_s + A_2\bar{R}_s) = (\widetilde{UCL}_P, \widetilde{UCL}_q, \widetilde{UCL}_r, \widetilde{UCL}_s)$$
$$\widetilde{CL}_{\bar{X}} = (\bar{\bar{X}}_p, \bar{\bar{X}}_q, \bar{\bar{X}}_r, \bar{\bar{X}}_s) = (\widetilde{CL}_P, \widetilde{CL}_q, \widetilde{CL}_r, \widetilde{CL}_s) \qquad (4)$$
$$\widetilde{LCL}_{\bar{X}} = \widetilde{CL} - A_2\bar{R} = (\bar{\bar{X}}_p, \bar{\bar{X}}_q, \bar{\bar{X}}_r, \bar{\bar{X}}_s) + A_2(\bar{R}_p, \bar{R}_q, \bar{R}_r, \bar{R}_s)$$
$$= (\bar{\bar{X}}_p - A_2\bar{R}_p, \bar{\bar{X}}_q - A_2\bar{R}_q, \bar{\bar{X}}_r - A_2\bar{R}_r, \bar{\bar{X}}_s - A_2\bar{R}_s) = (\widetilde{LCL}_P, \widetilde{LCL}_q, \widetilde{LCL}_r, \widetilde{LCL}_s)$$

The control limits for α-cut fuzzy $\tilde{\bar{\bar{X}}} - \tilde{R}$ control chart based on fuzzy trapezoidal number are

$$\widetilde{UCL}_{\bar{X}}^{\alpha} = (\bar{\bar{X}}_p^{\alpha}, \bar{\bar{X}}_q^{\alpha}, \bar{\bar{X}}_r^{\alpha}, \bar{\bar{X}}_s^{\alpha}) + A_2(\bar{R}_p^{\alpha}, \bar{R}_q^{\alpha}, \bar{R}_r^{\alpha}, \bar{R}_s^{\alpha})$$
$$= (\bar{\bar{X}}_p^{\alpha} + A_2\bar{R}_p^{\alpha}, \bar{\bar{X}}_q^{\alpha} + A_2\bar{R}_q^{\alpha}, \bar{\bar{X}}_r^{\alpha} + A_2\bar{R}_r^{\alpha}, \bar{\bar{X}}_s^{\alpha} + A_2\bar{R}_s^{\alpha}) = (\widetilde{UCL}_p^{\alpha}, \widetilde{UCL}_q^{\alpha}, \widetilde{UCL}_r^{\alpha}, \widetilde{UCL}_s^{\alpha})$$
$$\widetilde{CL}_{\bar{X}}^{\alpha} = (\bar{\bar{X}}_p^{\alpha}, \bar{\bar{X}}_q^{\alpha}, \bar{\bar{X}}_r^{\alpha}, \bar{\bar{X}}_s^{\alpha}) = (\widetilde{CL}_p^{\alpha}, \widetilde{CL}_q^{\alpha}, \widetilde{CL}_r^{\alpha}, \widetilde{CL}_s^{\alpha}) \qquad (5)$$
$$\widetilde{LCL}_{\bar{X}}^{\alpha} = (\bar{\bar{X}}_p^{\alpha}, \bar{\bar{X}}_q^{\alpha}, \bar{\bar{X}}_r^{\alpha}, \bar{\bar{X}}_s^{\alpha}) - A_2(\bar{R}_p^{\alpha}, \bar{R}_q^{\alpha}, \bar{R}_r^{\alpha}, \bar{R}_s^{\alpha})$$
$$= (\bar{\bar{X}}_p^{\alpha} - A_2\bar{R}_p^{\alpha}, \bar{\bar{X}}_q^{\alpha} - A_2\bar{R}_q^{\alpha}, \bar{\bar{X}}_r^{\alpha} - A_2\bar{R}_r^{\alpha}, \bar{\bar{X}}_s^{\alpha} - A_2\bar{R}_s^{\alpha}) = (\widetilde{LCL}_p^{\alpha}, \widetilde{LCL}_q^{\alpha}, \widetilde{LCL}_r^{\alpha}, \widetilde{LCL}_s^{\alpha})$$

where

$$\bar{\bar{X}}_p^{\alpha} = \bar{\bar{X}}_p + \alpha(\bar{\bar{X}}_q - \bar{\bar{X}}_p)$$
$$\bar{\bar{X}}_s^{\alpha} = \bar{\bar{X}}_s + \alpha(\bar{\bar{X}}_s - \bar{\bar{X}}_r) \qquad (6)$$
$$\bar{R}_p^{\alpha} = \bar{R}_p + \alpha(\bar{R}_q - \bar{R}_p)$$
$$\bar{R}_s^{\alpha} = \bar{R}_s + \alpha(\bar{R}_s - \bar{R}_r)$$

α - Level fuzzy midrange for α-cut fuzzy $\bar{\bar{X}}$ control chart based on ranges. These scalar values are the results obtained from transformation of fuzzy set back to its original value. The control limits and the centre line for the α - level fuzzy midrange for α - cut fuzzy $\bar{X}$ control chart based on ranges obtained is thereby used to monitor the production process and it can be obtained as given below

$$\widetilde{UCL}_{mr-\bar{X}}^{\alpha} = \widetilde{CL}_{mr-\bar{X}}^{\alpha} + A_2\left(\frac{\bar{R}_p^{\alpha} + \bar{R}_s^{\alpha}}{2}\right)$$
$$\widetilde{CL}_{mr-\bar{X}}^{\alpha} = f_{mr-\bar{X}}^{\alpha}(\widetilde{CL}_{\bar{X}}) = \frac{\bar{\bar{X}}_p^{\alpha} + \bar{\bar{X}}_s^{\alpha}}{2} \qquad (7)$$
$$\widetilde{LCL}_{mr-\bar{X}}^{\alpha} = \widetilde{CL}_{mr-\bar{X}}^{\alpha} - A_2\left(\frac{\bar{R}_p^{\alpha} + \bar{R}_s^{\alpha}}{2}\right)$$

Then for the k^{th} sample, the definition of α -level midranges for α -cut fuzzy $\bar{R}$ control chart is obtained as given below

$$s_{mr-\bar{X}.k}^{\alpha} = \left(\frac{(\bar{X}_{pk}+\bar{X}_{rk})+\alpha\{(\bar{X}_{qk}+\bar{X}_{pk})-(\bar{X}_{rk}+\bar{X}_{qk})\}}{2}\right) \qquad (8)$$

The condition for monitoring each sample can be defined as

$$\text{Process monitoring} = \begin{cases} \text{in control if} & \widetilde{UCL}_{mr-\bar{X}}^{\alpha} \le s_{mr-\bar{X}.k}^{\alpha} \le \widetilde{LCL}_{mr-\bar{X}}^{\alpha} \\ \text{out of control} & \text{otherwise} \end{cases} \qquad (9)$$

For the Fuzzy $\tilde{R}$ control chart:

The Shewhart R control chart limits are given below

$$UCL_R = D_4\bar{R}$$
$$CL_R = \bar{R} \qquad (10)$$
$$LCL_R = D_3\bar{R}$$

where D_4 and D_3 are control chart coefficient.

Hence the Fuzzy $\tilde{R}$ control chart with the control limits will be constructed following similar approach with the aid of trapezoidal fuzzy number

The control limits for α -level midranges for α -cut fuzzy $\tilde{R}$ control chart is obtained below.

$$\widetilde{UCL}^{\alpha}_{mr-R} = D_4 \widetilde{CL}^{\alpha}_{mr-R}$$

$$\widetilde{CL}^{\alpha}_{mr-R} = f^{\alpha}_{mr-\bar{X}}\left(\widetilde{CL}_{\bar{X}}\right) = \left(\frac{\bar{R}^{\alpha}_{p} + \bar{R}^{\alpha}_{s}}{2}\right) \quad (11)$$

$$\widetilde{LCL}^{\alpha}_{mr-R} = D_3 \widetilde{CL}^{\alpha}_{mr-R}$$

Then for the k^{th} sample, the definition of α -level midranges for α -cut fuzzy $\bar{R}$ control chart is obtained as given below

$$s^{\alpha}_{mr-R.j} = \left(\frac{(R_{pj} + R_{sj}) + \alpha\{(R_{qj} + R_{pk}) - (R_{sk} + R_{rk})\}}{2}\right) \quad (12)$$

The condition for monitoring each sample can be defined as Process monitoring

$$= \begin{cases} in\ control\ if & \widetilde{UCL}^{\alpha}_{mr-R} \le s^{\alpha}_{mr-R.j} \le \widetilde{LCL}^{\alpha}_{mr-R} \\ out\ of\ control & otherwise \end{cases} \quad (13)$$

IV. The Sensitivity Measure

Average Run Length (ARL) for fuzzy $\tilde{\bar{X}} - \tilde{S}$ and fuzzy $\tilde{\bar{X}} - \tilde{R}$ control chart

One of the measures of sensitivity of a control chart is the Average run length (ARL). It includes the in-control and out-of-control Average run length. The in-control Average run length is given by

$$ARL_0 = E(RL) = \frac{1}{\alpha} \quad (14)$$

where $\alpha = P(\bar{x} \notin [LCL, UCL]/IC)$

This indicates the probability of raising alarm in the process when the process is actually in-control.

The out-of-control Average run length

$$ARL_1 = E(RL) = \frac{1}{1-\beta} \quad (15)$$

where $\beta = P\{LCL \le \bar{x} \le UCL/\mu = \mu_0 + k\sigma\}$

This indicates the probability of type-two error, where k is the shift.

The data for this research was collected from measurements on piston rings inside diameter used by Pandurangan and Varadharajan (2011).

V. Results And Discussion

Table 1: Table of fuzzy $\tilde{\bar{X}} - \tilde{R}$ control chart and fuzzy $\tilde{R}$ control chart.

Sample (j)	$s^{0.65}_{mr-\bar{X}.j}$	Decision $73.9718 \le s^{0.65}_{mr-\bar{X}.j} \le 74.0166$	$s^{\alpha}_{mr-R.j}$	Decision $0 \le s^{\alpha}_{mr-R.j} \le 0.0820$
1	73.9965	In-Control	0.0435	In-Control
2	73.9943	In-Control	0.0400	In-Control
3	74.0016	In-Control	0.0399	In-Control
4	73.9936	In-Control	0.0367	In-Control
5	73.9989	In-Control	0.0421	In-Control
6	73.9908	In-Control	0.0374	In-Control
7	73.9940	In-Control	0.0325	In-Control
8	73.9925	In-Control	0.0441	In-Control
9	73.9977	In-Control	0.0355	In-Control
10	73.9907	In-Control	0.0412	In-Control
11	73.9946	In-Control	0.0375	In-Control
12	73.9943	In-Control	0.0359	In-Control
13	73.9891	In-Control	0.0395	In-Control
14	73.9897	In-Control	0.0448	In-Control
15	73.9952	In-Control	0.0314	In-Control

Table 2: Table of fuzzy $\tilde{\bar{\bar{X}}} - \tilde{S}$ control chart and fuzzy $\tilde{S}$ control chart.

Sample (j)	$s_{mr-\bar{X}.j}^{0.65}$	Decision $73.9800 \le s_{mr-\bar{X}.j}^{0.65} \le 74.0084$	$s_{mr-R.j}^{\alpha}$	Decision $0 \le s_{mr-R.j}^{\alpha} \le 0.02\ 08$
1	73.9965	In-Control	0.0045	In-Control
2	73.9943	In-Control	0.0052	In-Control
3	74.0016	In-Control	0.0038	In-Control
4	73.9936	In-Control	0.0048	In-Control
5	73.9989	In-Control	0.0061	In-Control
6	73.9908	In-Control	0.0045	In-Control
7	73.9940	In-Control	0.0051	In-Control
8	73.9925	In-Control	0.0055	In-Control
9	73.9977	In-Control	0.0057	In-Control
10	73.9907	In-Control	0.0048	In-Control
11	73.9946	In-Control	0.0039	In-Control
12	73.9943	In-Control	0.0069	In-Control
13	73.9891	In-Control	0.0049	In-Control
14	73.9897	In-Control	0.0054	In-Control
15	73.9952	In-Control	0.0041	In-Control

Table 3. Table of the Average Run Length (ARL) for fuzzy $\tilde{\bar{\bar{X}}} - \tilde{S}$ and $\tilde{\bar{\bar{X}}} - \tilde{R}$ control chart.

Shift (k)	Fuzzy $\tilde{\bar{\bar{X}}} - \tilde{S}$ Control Chart	Fuzzy $\tilde{\bar{\bar{X}}} - \tilde{R}$ Control Chart
0	370	370
0.1	293.3425	294.9467
0.2	176.1912	177.0468
0.3	98.91281	99.13952
0.4	56.38563	56.36660
0.5	33.39903	33.27250
0.6	20.56264	20.48872
0.7	13.21273	13.16807
0.8	8.855431	8.827546
0.9	6.184476	6.166513
1.0	4.495174	4.483266
1.1	3.395651	3.387547
1.2	2.661433	2.655790
1.3	2.160216	2.156211
1.4	1.811851	1.808966
1.5	1.566472	1.564373
1.6	1.392220	1.390686
1.7	1.268179	1.267060
1.8	1.187692	1.179402
1.9	1.121456	1.117883
2.0	1.078034	1.075430
2.5	1.005089	1.004779
3	1.000114	1.000103
3.5	1.000001	1.000001
4	1	1

From table 1 and table 2, it is clear that all the 15 sample points are within the control limits. Therefore, the control chart is said to be in-control.

VI. Conclusion

The study revealed that the fuzzy $\tilde{\bar{\bar{X}}} - \tilde{S}$ and $\tilde{\bar{\bar{X}}} - \tilde{R}$ are very sensitive to large process shift. As indicated from the results. When the shift is very large say 2 to 4, it takes just the first sample for fuzzy $\tilde{\bar{\bar{X}}} - \tilde{S}$ and $\tilde{\bar{\bar{X}}} - \tilde{R}$ to detect the shift with such magnitude. When at moderate shift say 1 to 1.9, it takes the fuzzy $\tilde{\bar{\bar{X}}} - \tilde{S}$ and $\tilde{\bar{\bar{X}}} - \tilde{R}$ approximately 4 to 2 samples to detect such shift. But when the shift is very small, the fuzzy $\tilde{\bar{\bar{X}}} - \tilde{S}$ and $\tilde{\bar{\bar{X}}} - \tilde{R}$, say 0.1, it takes the fuzzy $\tilde{\bar{\bar{X}}} - \tilde{S}$ and $\tilde{\bar{\bar{X}}} - \tilde{R}$ 293 and 294 samples to detect that the process mean has shifted by 0.1. These revealed that the fuzzy $\tilde{\bar{\bar{X}}} - \tilde{S}$ and $\tilde{\bar{\bar{X}}} - \tilde{R}$ is sensitive to large shift in the process mean but not sensitive to a small shift in the process mean.

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