Enhancement of Ratio Estimators for Finite Population Mean

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Abstract — **Hodges-Lehmann measures the median of the pairwise Walsh averages which is robust against outliers. A family of ratio-type estimators of finite population mean has been proposed. The study deals with the improvements on ratio estimators for the estimation of finite population mean of the variable of interest by using a known value of hodges-lehmann auxiliary information. The properties of the proposed estimators: bias and mean square error were obtained using Taylor's Series method. Empirical study is established in order to evaluate the performance or merit of the proposed estimators with respect to the conventional and other existing estimators. The results of the empirical study reveals that the proposed estimators are more efficient than the existing estimators.**

Keywords: *Study variable, Mean Square Error, Auxiliary information, Hodges-lehmann estimator, Population mean.*

I. Introduction

It is statistically proved and acceptable that appropriate use of auxiliary information in sample survey result to reduction of variance or mean square error of the estimators. The auxiliary variables are the population parameters such as populations' coefficient of variation, coefficient of kurtosis, skewness, median, decile, quartile, correlation, etc. Efficiency of estimators of the population parameters can be increased by the appropriate use of auxiliary information in relationship with auxiliary variable. Cochran [1] happened to be the first researcher that uses auxiliary variable and came up with what is known as ratio-type estimator for estimating population mean which is more efficient than the sample mean. Many authors have used different auxiliary variables in modifying estimators of population mean in return to increase the precision of the estimate by prior knowledge of population parameters. Researchers like Kadilar and Cingi [2 and 3] developed classes of ratio estimators for the estimation of population mean using known auxiliary information on coefficient of kurtosis, coefficient of variation and coefficient of variation. Abid *et al.* [4] also proposed set of ratio-type estimators for the population mean using non-conventional location parameters like mid-range, and tri-mean as auxiliary information. Other researchers are Koyuncu and Kadilar [5], Yan and Tian [6], Subramani and Kumarapandiyan [7], Jeelani *et al.* [8], Subzar *et al.* [9], and Abbas *et al*. [10].

In this study, we incorporated non-conventional location parameter which is highly robust against outliers in the work of Subzar *et al.* [11] whereas Subzar *et al.* [11] uses conventional location parameters. Our aim is to propose ratio mean estimators which can produce more efficient estimate of population mean.

Let $U=\{U_1,U_2,...,U_N\}$ be a finite population having N units and each $U_i=(X_i,\ Y_i),\ i=1,2,3,...,N$ has a pair of values. Y is the study variable and X is the auxiliary variable which is correlated with Y, where $y=\{y_1,y_2,...,y_n\}$ and $x=\{x_1,x_2,...,x_n\}$ are the n sample values. $\overline{y}$ and $\overline{x}$ are the sample means of the study and auxiliary variables respectively. S_y^2 and S_x^2 are the population mean squares of Y and X respectively and s_y^2 and s_x^2 be respective sample mean squares based on the random sample of size n drawn without replacement.

II. Literature Review

Cochran [1] proposed the ratio estimator for estimating population mean $\left(\bar{Y}\right)$ of the study variable $\left(Y\right)$ given as:

$$\hat{\bar{Y}}_r = \frac{\bar{y}}{\bar{x}}\bar{X} \quad (1)$$

$$Bias\left(\hat{\bar{Y}}\right) = \gamma\frac{1}{\bar{X}}\left(RS_x^2 - \rho S_x S_y\right) \quad (2)$$

$$MSE\left(\hat{\bar{Y}}_r\right) = \gamma\left(S_y^2 + R^2 S_x^2 - 2R\rho S_x S_y\right) \quad (3)$$

where $R = \dfrac{\bar{Y}}{\bar{X}}$

Abbas *et al.* [12] proposed a class of ratio type estimators for finite population mean using known values of decile mean $\left(DM\right)$, and coefficient of variation $\left(C_x\right)$ of auxiliary information, the biases, constants and mean square errors are given as:

$$\hat{\bar{Y}}_1 = \frac{\bar{y} + b\left(\bar{X} - \bar{x}\right)}{\left(\bar{x} + DM\right)}\left(\bar{X} + DM\right) \quad (4)$$

$$\hat{\bar{Y}}_2 = \frac{\bar{y} + b\left(\bar{X} - \bar{x}\right)}{\left(\bar{x}C_x + DM\right)}\left(\bar{X}C_x + DM\right) \quad (5)$$

$$\hat{\bar{Y}}_3 = \frac{\bar{y} + b\left(\bar{X} - \bar{x}\right)}{\left(\bar{x}\rho + DM\right)}\left(\bar{X}\rho + DM\right) \quad (6)$$

$$Bias\left(\hat{\bar{Y}}_i\right) = \gamma\frac{S_x^2}{\bar{Y}}R_i^2, \qquad where\ i = 1,2,3 \quad (9)$$

$$MSE\left(\hat{\bar{Y}}_i\right) = \gamma\left(R_i^2 S_x^2 + S_y^2\left(1 - \rho^2\right)\right) \qquad where\ i = 1,2,3 \quad (10)$$

$$R_1 = \frac{\bar{Y}}{\bar{X} + DM},\ R_2 = \frac{\bar{Y}C_x}{\bar{X}C_x + DM},\ R_3 = \frac{\bar{Y}\rho}{\bar{X}\rho + DM}$$

Subzar *et al.* [11] proposed a class ratio type estimators using the values of median $\left(M_d\right)$, quartile deviation $\left(QD\right)$, gini's mean difference $\left(G\right)$, downton's method $\left(D\right)$, probability weighted moments $\left(S_{pw}\right)$, and their linear combination with correlation coefficient $\left(\rho\right)$ and coefficient of variation $\left(C_x\right)$ of auxiliary information given as:

$$\hat{\bar{Y}}_4 = \frac{\bar{y} + b\left(\bar{X} - \bar{x}\right)}{\left(\bar{x} + \psi_1\right)}\left(\bar{X} + \psi_1\right) \quad (11)$$

$$\hat{\bar{Y}}_5 = \frac{\bar{y} + b\left(\bar{X} - \bar{x}\right)}{\left(\bar{x} + \psi_2\right)}\left(\bar{X} + \psi_2\right) \quad (12)$$

$$\hat{\bar{Y}}_6 = \frac{\bar{y} + b\left(\bar{X} - \bar{x}\right)}{\left(\bar{x} + \psi_3\right)}\left(\bar{X} + \psi_3\right) \quad (13)$$

$$\hat{\bar{Y}}_7 = \frac{\bar{y} + b\left(\bar{X} - \bar{x}\right)}{\left(\bar{x} + \psi_4\right)}\left(\bar{X} + \psi_4\right) \quad (14)$$

$$\hat{\bar{Y}}_8 = \frac{\bar{y} + b\left(\bar{X} - \bar{x}\right)}{\left(\bar{x} + \psi_5\right)}\left(\bar{X} + \psi_5\right) \quad (15)$$

$$\hat{\bar{Y}}_9 = \frac{\bar{y} + b\left(\bar{X} - \bar{x}\right)}{\left(\bar{x} + \psi_6\right)}\left(\bar{X} + \psi_6\right) \quad (16)$$

$$\hat{\bar{Y}}_{10} = \frac{\bar{y} + b\left(\bar{X} - \bar{x}\right)}{\left(\bar{x}\rho + \psi_1\right)}\left(\bar{X}\rho + \psi_1\right) \quad (17)$$

$$\hat{\bar{Y}}_{11} = \frac{\bar{y} + b\left(\bar{X} - \bar{x}\right)}{\left(\bar{x}\rho + \psi_2\right)}\left(\bar{X}\rho + \psi_2\right) \quad (18)$$

$$\hat{\bar{Y}}_{12} = \frac{\bar{y} + b\left(\bar{X} - \bar{x}\right)}{\left(\bar{x}\rho + \psi_3\right)}\left(\bar{X}\rho + \psi_3\right) \quad (19)$$

$$\hat{\bar{Y}}_{13} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}\rho+\psi_4\right)}\left(\bar{X}\rho+\psi_4\right) \quad (20)$$

$$\hat{\bar{Y}}_{14} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}\rho+\psi_5\right)}\left(\bar{X}\rho+\psi_5\right) \quad (21)$$

$$\hat{\bar{Y}}_{15} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}\rho+\psi_6\right)}\left(\bar{X}\rho+\psi_6\right) \quad (22)$$

$$\hat{\bar{Y}}_{16} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}C_x+\psi_1\right)}\left(\bar{X}C_x+\psi_1\right) \quad (23)$$

$$\hat{\bar{Y}}_{17} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}C_x+\psi_2\right)}\left(\bar{X}C_x+\psi_2\right) \quad (24)$$

$$\hat{\bar{Y}}_{18} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}C_x+\psi_3\right)}\left(\bar{X}C_x+\psi_3\right) \quad (25)$$

$$\hat{\bar{Y}}_{19} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}C_x+\psi_4\right)}\left(\bar{X}C_x+\psi_4\right) \quad (26)$$

$$\hat{\bar{Y}}_{20} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}C_x+\psi_5\right)}\left(\bar{X}C_x+\psi_5\right) \quad (27)$$

$$\hat{\bar{Y}}_{21} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}C_x+\psi_6\right)}\left(\bar{X}C_x+\psi_6\right) \quad (28)$$

where

$$\psi_1=\left(M_d+G\right),\ \psi_2=\left(M_d+D\right),\ \psi_3=\left(M_d+S_{pw}\right),\ \psi_4=\left(QD+G\right),\ \psi_5=\left(QD+D\right),\ \psi_6=\left(QD+S_{pw}\right)$$

$$Bias\left(\hat{\bar{Y}}_j\right)=\gamma\frac{S_x^2}{\bar{Y}}R_j^2, \qquad where\ \ j=4,5,\ldots,21 \quad (29)$$

$$MSE\left(\hat{\bar{Y}}_j\right)=\gamma\left(R_j^2S_x^2+S_y^2\left(1-\rho^2\right)\right) \qquad where\ \ j=4,5,\ldots,21 \quad (30)$$

$$R_4=\frac{\bar{Y}}{\bar{X}+\psi_1},\ R_5=\frac{\bar{Y}}{\bar{X}+\psi_2},\ R_6=\frac{\bar{Y}}{\bar{X}+\psi_3} R_7=\frac{\bar{Y}}{\bar{X}+\psi_4},\ R_8=\frac{\bar{Y}}{\bar{X}+\psi_5},\ R_9=\frac{\bar{Y}}{\bar{X}+\psi_6},$$

$$R_{10}=\frac{\bar{Y}\rho}{\bar{X}\rho+\psi_1},\ R_{11}=\frac{\bar{Y}\rho}{\bar{X}\rho+\psi_2},\ R_{12}=\frac{\bar{Y}\rho}{\bar{X}\rho+\psi_3} R_{13}=\frac{\bar{Y}\rho}{\bar{X}\rho+\psi_4},\ R_{14}=\frac{\bar{Y}\rho}{\bar{X}\rho+\psi_5},\ R_{15}=\frac{\bar{Y}\rho}{\bar{X}\rho+\psi_6}$$

$$R_{16}=\frac{\bar{Y}C_x}{\bar{X}C_x+\psi_1},\ R_{17}=\frac{\bar{Y}C_x}{\bar{X}C_x+\psi_2},\ R_{18}=\frac{\bar{Y}C_x}{\bar{X}C_x+\psi_3} R_{19}=\frac{\bar{Y}C_x}{\bar{X}C_x+\psi_4},\ R_{20}=\frac{\bar{Y}C_x}{\bar{X}C_x+\psi_5},\ R_{21}=\frac{\bar{Y}C_x}{\bar{X}C_x+\psi_6}$$

III. Proposed Estimators

Motivated by the work of Subzar *et al.* [11], we proposed ratio-type estimators for estimating population mean using know value of hodges-lehmann as:

$$\hat{\bar{Y}}_{p1} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}+\psi_1^*\right)}\left(\bar{X}+\psi_1^*\right) \quad (31)$$

$$\hat{\bar{Y}}_{p2} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}+\psi_2^*\right)}\left(\bar{X}+\psi_2^*\right) \quad (32)$$

$$\hat{\bar{Y}}_{p3} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}+\psi_3^*\right)}\left(\bar{X}+\psi_3^*\right) \quad (33)$$

$$\hat{\bar{Y}}_{p4} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}+\psi_4^*\right)}\left(\bar{X}+\psi_4^*\right) \quad (34)$$

$$\hat{\bar{Y}}_{p5} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}+\psi_5^*\right)}\left(\bar{X}+\psi_5^*\right) \quad (35)$$

$$\hat{\bar{Y}}_{p6} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}+\psi_6^*\right)}\left(\bar{X}+\psi_6^*\right) \quad (36)$$

$$\hat{\bar{Y}}_{p7} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}\rho+\psi_1^*\right)}\left(\bar{X}\rho+\psi_1^*\right) \quad (37)$$

$$\hat{\bar{Y}}_{p8} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}\rho+\psi_2^*\right)}\left(\bar{X}\rho+\psi_2^*\right) \quad (38)$$

$$\hat{\bar{Y}}_{p9} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}\rho+\psi_3^*\right)}\left(\bar{X}\rho+\psi_3^*\right) \quad (39)$$

$$\hat{\bar{Y}}_{p10} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}\rho+\psi_4^*\right)}\left(\bar{X}\rho+\psi_4^*\right) \quad (40)$$

$$\hat{\bar{Y}}_{p11} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}\rho+\psi_5^*\right)}\left(\bar{X}\rho+\psi_5^*\right) \quad (41)$$

$$\hat{\bar{Y}}_{p12} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}\rho+\psi_6^*\right)}\left(\bar{X}\rho+\psi_6^*\right) \quad (42)$$

$$\hat{\bar{Y}}_{p13} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}C_x+\psi_1^*\right)}\left(\bar{X}C_x+\psi_1^*\right) \quad (43)$$

$$\hat{\bar{Y}}_{p14} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}C_x+\psi_2^*\right)}\left(\bar{X}C_x+\psi_2^*\right) \quad (44)$$

$$\hat{\bar{Y}}_{p15} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}C_x+\psi_3^*\right)}\left(\bar{X}C_x+\psi_3^*\right) \quad (45)$$

$$\hat{\bar{Y}}_{p16} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}C_x+\psi_4^*\right)}\left(\bar{X}C_x+\psi_4^*\right) \quad (46)$$

$$\hat{\bar{Y}}_{p17} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}C_x+\psi_5^*\right)}\left(\bar{X}C_x+\psi_5^*\right) \quad (47)$$

$$\hat{\bar{Y}}_{p18} = \frac{\bar{y}+b\left(\bar{X}-\bar{x}\right)}{\left(\bar{x}C_x+\psi_6^*\right)}\left(\bar{X}C_x+\psi_6^*\right) \quad (48)$$

where

$\psi_1^*=\left(M_d+G\times HL\right), \psi_2^*=\left(M_d+D\times HL\right), \psi_3^*=\left(M_d+S_{pw}\times HL\right), \psi_4^*=\left(QD+G\times HL\right),$

$\psi_5^*=\left(QD+D\times HL\right), \psi_6^*=\left(QD+S_{pw}\times HL\right)$

In order to derive the bias and MSE, we define

$$\left.\begin{array}{l} Var\left(\bar{x}\right)=S_x^2,\ Var\left(\bar{y}\right)=S_y^2,\ Cov\left(\bar{x},\bar{y}\right)=\rho S_{yx}=\rho S_y S_x \\ \gamma=\frac{1-f}{n},\ f=\frac{n}{N} \end{array}\right\} \quad (49)$$

$$h\left(\bar{x},\bar{y}\right)=h\left(\bar{X},\bar{Y}\right)+\frac{\partial h\left(c,d\right)}{\partial c}\bigg|_{\bar{X},\bar{Y}}\left(\bar{x}-\bar{X}\right)+\frac{\partial h\left(c,d\right)}{\partial d}\bigg|_{\bar{X},\bar{Y}}\left(\bar{y}-\bar{Y}\right) \quad (50)$$

$$h\left(\bar{x},\bar{y}\right)=\hat{R}_{p1} \ \text{ and } \ h\left(\bar{X},\bar{Y}\right)=R \quad (51)$$

$$\hat{R}_{p1}-R=h\left(\bar{X},\bar{Y}\right)+\frac{\partial\left(\bar{y}+b\left(\bar{x}-\bar{X}\right)/\left(\bar{X}+\psi_1^*\right)\right)}{\partial\bar{x}}\bigg|_{\bar{X},\bar{Y}}\left(\bar{x}-\bar{X}\right)+\frac{\partial\left(\bar{y}+b\left(\bar{x}-\bar{X}\right)/\left(\bar{X}+\psi_1^*\right)\right)}{\partial\bar{y}}\bigg|_{\bar{X},\bar{Y}}\left(\bar{y}-\bar{Y}\right) \quad (52)$$

$$\hat{R}_{p1}-R=-\left(\frac{\bar{y}}{\left(\bar{x}+\psi_1^*\right)^2}+\frac{b\left(\bar{X}+\psi_1^*\right)}{\left(\bar{x}+\psi_1^*\right)^2}\right)|_{\bar{X},\bar{Y}}\left(\bar{x}-\bar{X}\right)+\frac{1}{\left(\bar{x}+\psi_1^*\right)}|_{\bar{X},\bar{Y}}\left(\bar{y}-\bar{Y}\right) \quad (53)$$

Squaring and taking expectation of (53), gives

$$E\left(\hat{R}_{p1}-R\right)^2=E\left(-\left(\frac{\bar{y}}{\left(\bar{X}+\psi_1^*\right)^2}+\frac{b\left(\bar{X}+\psi_1^*\right)}{\left(\bar{X}+\psi_1^*\right)^2}\right)|_{\bar{X},\bar{Y}}\left(\bar{x}-\bar{X}\right)+\frac{1}{\left(\bar{X}+\psi_1^*\right)}|_{\bar{X},\bar{Y}}\left(\bar{y}-\bar{Y}\right)\right)^2 \quad (54)$$

Expanding (54), we have

$$E\left(\hat{R}_{p1}-R\right)^{2}=\frac{\left(\bar{Y}+B\left(\bar{X}+\psi_{1}^{*}\right)\right)^{2}}{\left(\bar{X}+\psi_{1}^{*}\right)^{4}}Var\left(\bar{x}\right)-\frac{2\left(\bar{Y}+B\left(\bar{X}+\psi_{1}^{*}\right)\right)}{\left(\bar{X}+\psi_{1}^{*}\right)^{3}}Cov\left(\bar{x},\bar{y}\right)+\frac{1}{\left(\bar{X}+\psi_{1}^{*}\right)^{2}}Var\left(\bar{y}\right) \quad (55)$$

Factorizing (55) give (56)

$$E\left(\hat{R}_{p1}-R\right)^{2}=\frac{1}{\left(\bar{x}+\psi_{1}^{*}\right)^{2}}\left[\frac{\left(\bar{Y}+B\left(\bar{X}+\psi_{1}^{*}\right)\right)^{2}}{\left(\bar{X}+\psi_{1}^{*}\right)^{2}}Var\left(\bar{x}\right)-\frac{2\left(\bar{Y}+B\left(\bar{X}+\psi_{1}^{*}\right)\right)}{\left(\bar{X}+\psi_{1}^{*}\right)}Cov\left(\bar{x},\bar{y}\right)+Var\left(\bar{y}\right)\right] \quad (56)$$

Recall that $B=\frac{S_{yx}}{S_{x}^{2}}=\frac{\rho S_{y}S_{x}}{S_{x}^{2}}=\frac{\rho S_{y}}{S_{x}}$ (57)

$$MSE\left(\hat{\bar{Y}}_{p1}\right)=\left(\bar{X}+\psi_{1}^{*}\right)^{2}E\left(\hat{R}_{p1}-R\right)^{2} \quad (58)$$

Substituting (56) in (58), gives (59)

$$MSE\left(\hat{\bar{Y}}_{p1}\right)=\frac{\left(\bar{Y}+B\left(\bar{X}+\psi_{1}^{*}\right)\right)^{2}}{\left(\bar{X}+\psi_{1}^{*}\right)^{2}}Var\left(\bar{x}\right)-\frac{2\left(\bar{Y}+B\left(\bar{X}+\psi_{1}^{*}\right)\right)}{\left(\bar{X}+\psi_{1}^{*}\right)}Cov\left(\bar{x},\bar{y}\right)+Var\left(\bar{y}\right) \quad (59)$$

$$MSE\left(\hat{\bar{Y}}_{p1}\right)=\frac{\bar{Y}^{2}+B^{2}\left(\bar{X}+\psi_{1}^{*}\right)^{2}+2B\left(\bar{X}+\psi_{1}^{*}\right)\bar{Y}}{\left(\bar{X}+\psi_{1}^{*}\right)^{2}}Var\left(\bar{x}\right)-\frac{2\left(\bar{Y}+B\left(\bar{X}+\psi_{1}^{*}\right)\right)}{\left(\bar{X}+\psi_{1}^{*}\right)}Cov\left(\bar{x},\bar{y}\right)+Var\left(\bar{y}\right) \quad (60)$$

$$MSE\left(\hat{\bar{Y}}_{p1}\right)=\gamma\left(\frac{\bar{Y}^{2}}{\left(\bar{X}+\psi_{1}^{*}\right)^{2}}+B^{2}S_{x}^{2}+\frac{2B\bar{Y}}{\left(\bar{X}+\psi_{1}^{*}\right)}-\left(\frac{2\bar{Y}}{\left(\bar{X}+\psi_{1}^{*}\right)}+2B\right)S_{yx}+S_{y}^{2}\right) \quad (61)$$

$$MSE\left(\hat{\bar{Y}}_{p1}\right)=\gamma\left(R_{p1}^{2}S_{x}^{2}+B^{2}S_{x}^{2}+2BR_{p1}S_{x}^{2}-2R_{p1}S_{yx}-2BS_{yx}+S_{y}^{2}\right),\quad R_{p1}=\frac{\bar{Y}}{\bar{X}+\psi_{1}^{*}} \quad (62)$$

Substituting (56) in (62), gives (63) as

$$MSE\left(\hat{\bar{Y}}_{p1}\right)=\gamma\left(R_{p1}^{2}S_{x}^{2}+\rho^{2}S_{y}^{2}+2R_{p1}\rho S_{y}S_{x}-2R_{p1}\rho S_{y}S_{x}-2\rho^{2}S_{y}^{2}+S_{y}^{2}\right) \quad (63)$$

$$MSE\left(\hat{\bar{Y}}_{p1}\right)=\gamma\left(R_{p1}^{2}S_{x}^{2}+S_{y}^{2}\left(1-\rho^{2}\right)\right) \quad (64)$$

So, the bias is derived as:

$$Bias\left(\hat{\bar{Y}}_{Pi}\right)=\gamma\frac{S_{x}^{2}}{\bar{Y}}R_{Pi}^{2},\qquad \left(i=1,2,\ldots,18\right) \quad (65)$$

For the remaining proposed estimators, we have

$$MSE\left(\hat{\bar{Y}}_{Pi}\right)=\gamma\left(R_{Pi}^{2}S_{x}^{2}+S_{y}^{2}\left(1-\rho^{2}\right)\right),\qquad \left(i=1,2,\ldots,18\right) \quad (66)$$

where

$$R_{p1}=\frac{\bar{Y}}{\bar{X}+\psi_{1}^{*}},\ R_{p2}=\frac{\bar{Y}}{\bar{X}+\psi_{2}^{*}},\ R_{p3}=\frac{\bar{Y}}{\bar{X}+\psi_{3}^{*}}\ R_{p4}=\frac{\bar{Y}}{\bar{X}+\psi_{4}^{*}},\ R_{p5}=\frac{\bar{Y}}{\bar{X}+\psi_{5}^{*}},\ R_{p6}=\frac{\bar{Y}}{\bar{X}+\psi_{6}^{*}},$$

$$R_{p7}=\frac{\bar{Y}\rho}{\bar{X}\rho+\psi_1^*},\ R_{p8}=\frac{\bar{Y}\rho}{\bar{X}\rho+\psi_2^*},\ R_{p9}=\frac{\bar{Y}\rho}{\bar{X}\rho+\psi_3^*}\ R_{p10}=\frac{\bar{Y}\rho}{\bar{X}\rho+\psi_4^*},\ R_{p11}=\frac{\bar{Y}\rho}{\bar{X}\rho+\psi_5^*},\ R_{p12}=\frac{\bar{Y}\rho}{\bar{X}\rho+\psi_6^*}$$

$$R_{p13}=\frac{\bar{Y}C_x}{\bar{X}C_x+\psi_1^*},\ R_{p14}=\frac{\bar{Y}C_x}{\bar{X}C_x+\psi_2^*},\ R_{p15}=\frac{\bar{Y}C_x}{\bar{X}C_x+\psi_3^*},\ R_{p16}=\frac{\bar{Y}C_x}{\bar{X}C_x+\psi_4^*},\ R_{p17}=\frac{\bar{Y}C_x}{\bar{X}C_x+\psi_5^*},\ R_{p18}=\frac{\bar{Y}C_x}{\bar{X}C_x+\psi_6^*}$$

3.1 Efficiency Comparisons

Efficiencies of the proposed estimators are compared with efficiencies of the existing estimators in the study

The $\hat{\bar{Y}}_{pi}-family$ of estimators of the finite population mean is more efficient than $\hat{\bar{Y}}_r$ if,

$$MSE\left(\hat{\bar{Y}}_{pi}\right)< MSE\left(\hat{\bar{Y}}_r\right)\qquad i=1,2,\ldots,18$$

$$\left(R_{pi}^2S_x^2+S_y^2\left(1-\rho^2\right)\right)<\left(S_y^2+R^2S_x^2-2R\rho S_xS_y\right)\quad(67)$$

The $\hat{\bar{Y}}_{pi}-family$ of proposed estimators of the population mean is more efficient than $\hat{\bar{Y}}_j$ if,

$$MSE\left(\hat{\bar{Y}}_{pi}\right)< MSE\left(\hat{\bar{Y}}_j\right)\qquad i=1,2,\ldots,18\quad j=1,2,3,\ldots,21$$

$$\left(R_{pi}^2S_x^2+S_y^2\left(1-\rho^2\right)\right)<\left(R_j^2S_x^2+S_y^2\left(1-\rho^2\right)\right)\quad(68)$$

When conditions (67), and (68) are satisfied, we can conclude that the proposed estimators are more efficient than the existing estimators.

3.2 Numerical Illustration

In order to evaluate the performance of the proposed estimator, we considered three natural populations I, II, and III as follows:

Table 1: Characteristics of Populations I, II, and III (Subzar *et al.* [11])

Parameter	Population I	Population II	Population III
N	34	34	80
n	20	20	20
$\bar{Y}$	856.4117	856.4117	5182.637
$\bar{X}$	199.4412	208.8823	1126.463
ρ	0.4453	0.4491	0.941
S_y	733.1407	733.1407	1835.659
C_y	0.8561	0.8561	0.354193
S_x	150.2150	150.5059	845.610
C_x	0.7531	0.7205	0.7506772
β_2	1.0445	0.0978	-0.063386
β_1	1.1823	0.9782	1.050002
M_d	142.5	150	757.5
MR	320	284.5	1795.5
TM	165.562	162.25	931.562
HL	320	190	1040.5
QD	184	80,25	588.125
G	162.996	155.446	901.081
D	144.481	140.891	801.381
S_{pw}	206.944	199.961	791.364
DM	206.944	234.82	1150.7

Table1 shows the population parameters of auxiliary variable.

Table 2: Constant and Bias of Selected Existing and Proposed Estimators

Estimator	Constant			Bias		
	Pop-I	Pop-II	Pop-III	Pop-I	Pop-II	Pop-III
$\hat{Y}_r$	4.294	4.100	4.601	4.940	4.270	60.877
$\hat{Y}_1$	2.107	1.9301	2.276	2.137	2.2087	26.800
$\hat{Y}_2$	1.806	1.6013	1.949	1.483	1.3964	19.650
$\hat{Y}_3$	1.289	1.1703	2.206	0.800	0.7459	25.188
$\hat{Y}_4$	1.6960	1.6651	1.9813	1.5604	1.5098	20.312
$\hat{Y}_5$	1.7606	1.7136	2.0598	1.6815	1.599	21.953
$\hat{Y}_6$	1.5602	1.5324	2.0681	1.3205	1.2788	22.129
$\hat{Y}_7$	1.8955	1.9263	1.8608	1.9490	2.0207	17.916
$\hat{Y}_8$	1.9764	1.9915	1.9299	2.1191	1.9915	2.1598
$\hat{Y}_9$	1.7274	1.751	1.9371	1.6187	1.6696	19.416
$\hat{Y}_{10}$	0.9671	0.9633	1.7938	0.5074	0.5053	16.650
$\hat{Y}_{11}$	1.0148	0.9997	1.8622	0.5586	0.5443	17.942
$\hat{Y}_{12}$	0.8701	0.8666	1.8693	0.4107	0.409	18.079
$\hat{Y}_{13}$	1.1177	1.1672	1.9130	0.6777	0.7419	18.936
$\hat{Y}_{14}$	1.1818	1.2211	1.9909	0.7577	0.8121	20.508
$\hat{Y}_{15}$	0.9902	1.0283	1.9991	0.5318	0.5758	20.677
$\hat{Y}_{16}$	1.4153	1.3533	1.5540	1.0866	0.9973	12.495
$\hat{Y}_{17}$	1.4752	1.3979	1.6184	1.1806	1.0642	13.552
$\hat{Y}_{18}$	1.2908	1.2329	1.6252	0.9038	0.8278	13.666
$\hat{Y}_{19}$	1.6021	1.5977	1.6667	1.3923	1.3901	14.373
$\hat{Y}_{20}$	1.6793	1.6603	1.7410	1.5298	1.5011	15.684
$\hat{Y}_{21}$	1.4444	1.4326	1.7489	1.1317	1.1176	15.825

Table 2 : Continued

Estimator	Constant			Bias		
	Pop-I	Pop-II	Pop-III	Pop-I	Pop-II	Pop-III
$\hat{\bar{Y}}_{p1}$	0.0282	0.0004	0.0055	0.0004	0.0286	0.00016
$\hat{\bar{Y}}_{p2}$	0.0318	0.0005	0.0062	0.0005	0.0316	0.00020
$\hat{\bar{Y}}_{p3}$	0.0223	0.0003	0.0063	0.0003	0.0223	0.00020
$\hat{\bar{Y}}_{p4}$	0.0282	0.0004	0.0055	0.0004	0.0282	0.00016
$\hat{\bar{Y}}_{p5}$	0.0318	0.0005	0.0062	0.0005	0.0311	0.00020
$\hat{\bar{Y}}_{p6}$	0.0223	0.0003	0.0063	0.0003	0.0221	0.00020
$\hat{\bar{Y}}_{p7}$	0.0283	0.0005	0.0055	0.0004	0.0288	0.00016
$\hat{\bar{Y}}_{p8}$	0.0319	0.0005	0.0062	0.0006	0.0317	0.00020
$\hat{\bar{Y}}_{p9}$	0.0224	0.0003	0.0063	0.0003	0.0224	0.00020
$\hat{\bar{Y}}_{p10}$	0.0283	0.0004	0.0055	0.0004	0.0283	0.00016
$\hat{\bar{Y}}_{p11}$	0.0319	0.0005	0.0062	0.0006	0.0312	0.00020
$\hat{\bar{Y}}_{p12}$	0.0223	0.0003	0.0063	0.0003	0.0221	0.00020
$\hat{\bar{Y}}_{p13}$	0.0283	0.0004	0.0055	0.0004	0.0287	0.00016
$\hat{\bar{Y}}_{p14}$	0.0319	0.0005	0.0062	0.0006	0.0316	0.00020
$\hat{\bar{Y}}_{p15}$	0.0223	0.0003	0.0063	0.0003	0.0224	0.00020
$\hat{\bar{Y}}_{p16}$	0.0282	0.0004	0.0055	0.0004	0.0283	0.00016
$\hat{\bar{Y}}_{p17}$	0.0318	0.0005	0.0062	0.0005	0.0311	0.00020
$\hat{\bar{Y}}_{p18}$	0.0223	0.0003	0.0063	0.0003	0.0221	0.00020

Table 2 shows the constant and bias of existing and proposed estimators for the three Populations. The result revealed that the proposed estimators have least constant and bias compared to the existing estimators. This implies that the proposed estimators are proficient than the existing selected estimators.

Table 3: The Mean Square Error (MSE) and the Percentage Relative Efficiency (PRE) of the Existing and Proposed Estimators.

Estimator	MSE			PRE		
	Pop-I	Pop-II	Pop-III	Pop-I	Pop-II	Pop-III
$\hat{\bar{Y}}_r$	10960.76	10539.27	189775.1	100	100	100
$\hat{\bar{Y}}_1$	10934.74	10571.58	153292.6	100.2380	99.69437	123.7993
$\hat{\bar{Y}}_2$	10386.83	10030.11	116239.3	105.5256	105.0763	163.2624
$\hat{\bar{Y}}_3$	9644.04	9472.95	144936.7	113.6532	111.2565	130.9365
$\hat{\bar{Y}}_4$	10208.16	10333.07	119741.5	107.3725	101.9955	158.4873
$\hat{\bar{Y}}_5$	10311.83	10203.59	128249.9	106.2931	103.2898	147.9729
$\hat{\bar{Y}}_6$	10002.72	9929.39	129161.3	109.5778	106.1422	146.9288
$\hat{\bar{Y}}_7$	10540.91	10564.74	107326.5	103.9831	99.75892	176.8204
$\hat{\bar{Y}}_8$	10686.6	10683.87	114349.5	102.5655	98.64656	165.9606
$\hat{\bar{Y}}_9$	10258.09	10264.06	115098.9	106.8499	102.6813	164.8800
$\hat{\bar{Y}}_{10}$	9306.32	9266.94	100762.1	117.7776	113.7298	188.3398
$\hat{\bar{Y}}_{11}$	9350.19	9300.31	107457.3	117.2250	113.3217	176.6051
$\hat{\bar{Y}}_{12}$	9223.53	9184.47	108172.8	118.8348	114.7510	175.4370
$\hat{\bar{Y}}_{13}$	9452.18	9469.56	112609.8	115.9601	111.2963	168.5245
$\hat{\bar{Y}}_{14}$	9250.7	9529.64	120761.3	118.4857	110.5946	157.1489
$\hat{\bar{Y}}_{15}$	9327.27	9327.31	121636.1	117.5131	112.9937	156.0187
$\hat{\bar{Y}}_{16}$	9802.37	9688.30	79228.58	111.8174	108.7835	239.5286
$\hat{\bar{Y}}_{17}$	9882.86	9745.56	84709.31	110.9068	108.1443	224.0310
$\hat{\bar{Y}}_{18}$	9645.86	9543.10	85298.12	113.6318	110.4386	222.4845
$\hat{\bar{Y}}_{19}$	10064.19	10024.69	88963.27	108.9085	105.1331	213.3185
$\hat{\bar{Y}}_{20}$	10181.93	10119.77	95756.01	107.6491	104.1454	198.1861
$\hat{\bar{Y}}_{21}$	9841.01	9791.32	96489.29	111.3784	107.6389	196.6800
$\hat{\bar{Y}}_{p1}$	8872.134	8834.536	14471.63	123.5414	119.2962	1311.360

$\hat{\bar{Y}}_{p2}$	8872.233	8834.618	14471.84	123.5400	119.2951	1311.341
$\hat{\bar{Y}}_{p3}$	8871.994	8834.385	14471.87	123.5434	119.2983	1311.338
$\hat{\bar{Y}}_{p4}$	8872.133	8834.525	14471.63	123.5414	119.2964	1311.360
$\hat{\bar{Y}}_{p5}$	8872.232	8834.603	14471.84	123.5401	119.2953	1311.341
$\hat{\bar{Y}}_{p6}$	8871.994	8834.38	14471.87	123.5434	119.2984	1311.338
$\hat{\bar{Y}}_{p7}$	8872.136	8834.539	14471.63	123.5414	119.2962	1311.360
$\hat{\bar{Y}}_{p8}$	8872.237	8834.622	14471.84	123.54	119.2951	1311.341
$\hat{\bar{Y}}_{p9}$	8871.996	8834.387	14471.87	123.5433	119.2983	1311.338
$\hat{\bar{Y}}_{p10}$	8872.135	8834.528	14471.63	123.5414	119.2964	1311.360
$\hat{\bar{Y}}_{p11}$	8872.236	8834.607	14471.84	123.54	119.2953	1311.341
$\hat{\bar{Y}}_{p12}$	8871.995	8834.382	14471.87	123.5434	119.2983	1311.338
$\hat{\bar{Y}}_{p13}$	8872.135	8834.537	14471.63	123.5414	119.2962	1311.360
$\hat{\bar{Y}}_{p14}$	8872.235	8834.62	14471.84	123.54	119.2951	1311.341
$\hat{\bar{Y}}_{p15}$	8871.995	8834.386	14471.87	123.5434	119.2983	1311.338
$\hat{\bar{Y}}_{p16}$	8872.134	8834.526	14471.63	123.5414	119.2964	1311.360
$\hat{\bar{Y}}_{p17}$	8872.234	8834.605	14471.84	123.54	119.2953	1311.341
$\hat{\bar{Y}}_{p18}$	8871.994	8834.381	14471.87	123.5434	119.2983	1311.338

Table 3 shows the mean square error (MSE) and percentage relative efficiency (PRE) of existing and proposed estimators for the three Populations. The result revealed that the proposed estimators have least mean square error (MSE) and highest percentage relative efficiency (PRE) compared to the existing estimators. This implies that the proposed estimators are more efficient than the existing selected estimators

IV. Results And Discussion

A family of efficient ratio estimators of population mean is proposed using the auxiliary information of non-conventional location parameter and the performance of the proposed estimators over the existing estimators were established. Tables 2-3 show the value of Constants, Biases, Mean Square Errors (MSE) and Percentage Relative Efficiency (PRE) of the existing and proposed estimators of all the three populations used in the study. The outcome revealed that the proposed ratio estimators has the least MSE and higher PRE compare to the conventional and other existing estimators considered in the study.

V. Conclusion

The result in Table 3 revealed that the proposed estimators using the auxiliary information of non-conventional location parameter performed better than the existing estimators considered in the study. It is clear that the proposed estimators do better than the existing estimators having minimum Mean Square Error (MSE) and the highest Percentage Relative Efficiency (PRE). We therefore recommend the proposed estimators for use in practical applications for estimating population mean.

References

[1] Cochran W.G. (1940). *Sampling Techniques, 3rd Edition.* Wiley Eastern Limited, New Delhi India.

[2] Kadilar, C. and Cingi, H. (2004). Ratio Estimators in Simple Random Sampling. *Applied Mathematics and Computation,* 151, pp. 893 – 902.

[3] Kadilar C. and Cingi, H. (2006). An Improvement in estimating the population mean by using the correlation coefficient. *Hacettepe Journal of Mathematics and Statistics,* 35, pp.103 – 109.

[4] Abid, M., Abbas, N. Nazir, H. Z., Lin, Z. (2016). Enhancing the Mean Ratio Estimators for Estimating Population Mean Using Non-conventional Location Parameters. *Revista Colombiana de Estadistica*, 39(1), pp. 63 – 79.

[5] Koyuncu, N. and Kadilar, C. (2009). Efficient Estimators for the Population Mean, Hacettepe Journal of Mathematics and Statistics, 38(2), pp. 217-225.

[6] Yan, Z. and Tian, B. (2010). Ratio Method to the Mean Estimation using Coefficient of Skewness of Auxiliary Variable. *In: Information Computing and Applications. ICICA.*

[7] Subramani, J. and Kumarapandiyan, G. (2012). Estimation of Population Mean using Co-efficient of Variation and Median of an Auxiliary Variable. *International Journal of Probability and Statistics,* 1, pp. 111 – 118.

[8] Jeelani, M.I., Maqbool S. & Mir S.A. (2013). Modified Ratio Estimators of Population Mean using Linear Combination of Estimators of Population Mean using Linear Combination of Coefficient of Skewness and Quartile Deviation. *International Journal of Modern Mathematical Sciences,* 6, pp.174 – 183.

[9] Subzar, M., Raja, S.M., Rishu, J. and Shabeer, M. (2017). Enhancing Ratio Type Estimators for Estimating Population Mean using Auxiliary Information in Survey Sampling. *International Journal of Agricultural Statistical Sciences*, 13(1), pp.213-220.

[10] Nasir, A. Muhammad, A. Muhammad, T. Nasir, A. and Zawar, H. (2018). Enhancing Ratio Estimators for Estimating Population Mean using Maximum value of Auxiliary Variable. *Journal of the National Science Foundation of Sri Lanka*, 46(3), pp. 453 – 463.

[11] Subzar, M. Maqbool, S. Raja, T. A., Surya K. P. and Sharma, P. (2018). Efficient Estimators of Population Mean Using Auxiliary Information Under Simple Random Sampling. *Statistics in Transition new series*, 19(2), pp. 1 – 20.

APPENDIX

Notations

The following are the other notations used in this study.

N : Population size
n : Sample size
Y : Study variable
X : Auxiliary variable
$\bar{y}, \bar{x}$: Sample means of study and auxiliary variables
$\bar{Y}, \bar{X}$: Population means of study and auxiliary variables
ρ : Coefficient of correlation
C_y, C_x : Coefficient of variations of study and auxiliary variables
Q_3 : The upper quartile
QD : Population Quartile Deviation
β_1 : Coefficient of skewness of auxiliary variable
β_2 : Coefficient of kurtosis of auxiliary variable
TM : Tri-Mean
M_d : Median of the auxiliary
MR : Population mid-range
HL : Hodges-Lehman estimator
G : Gini's Mean Difference
D : Downton's Method
S_{pw} : Probability weighted moments

$$\bar{X}=\frac{1}{N}\sum_{i=1}^{N}X_i,\quad \bar{Y}=\frac{1}{N}\sum_{i=1}^{N}Y_i,\quad \bar{x}=\frac{1}{n}\sum_{i=1}^{n}x_i,\quad \bar{y}=\frac{1}{n}\sum_{i=1}^{n}y_i,\quad \gamma=\frac{1-f}{n},$$

$$TM=\frac{(Q_1+2Q_2+Q_3)}{4},s_y^2=\frac{1}{n-1}\sum_{i=1}^{n}(y_i-\bar{y})^2,\quad s_x^2=\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2,\ S_y^2=\frac{1}{N-1}\sum_{i=1}^{N}(Y_i-\bar{Y})^2,$$

$$S_x^2=\frac{1}{N-1}\sum_{i=1}^{N}(X_i-\bar{X})^2,\ MR=\frac{X_{(1)}+X_{(N)}}{2},\ HL=Median\left(\frac{(X_i+X_j)}{2},1\le i\le j\le N\right)$$

$$G=\frac{4}{N-1}\sum_{i=1}^{N}\left(\frac{2i-N-1}{2N}\right)X_i,\ D=\frac{2\sqrt{\pi}}{N(N-1)}\sum_{i=1}^{N}\left(i-\frac{N+1}{2}\right)X_i,\ S_{pw}=\frac{2\sqrt{\pi}}{N^2}\sum_{i=1}^{N}(2i-N-1)X_i$$

$$QD=\frac{Q_3-Q_1}{2}.$$

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