On the Development of Four-Parameters Exponentiated Generalized Exponential Distribution

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***Abstract* — In this paper, a four parameter Exponentiated Generalized Exponentiated exponential distribution is derived from Exponentiated Generalized Family (EGF) of distribution. Some properties of the distribution are studied. The distribution is found to be unimodal and has a decreasing and increasing hazard rate depending on the shape parameters. The expressions for the moment, median, quantile, mean deviation, median deviation, skewness, kurtosis, Renyi entropy are obtained. Some known continuous distributions are special cases of the new derived distributions. Simulation study, maximum likelihood estimator and real life application of the model to data, shows that new distribution fits better than it's sub-models.**

Keywords - *Moment, Hazard rate, Kurtosis, Renyi Entropy, Unimodal, skewness, quantile function..*

I. Introduction

The exponential distribution (ED) also known as negative exponential distribution is a probability distribution that describes the time between event in a Poisson point process i.e. a process in which event occurs continuously and independently at a constant average rate. The ED is a very popular statistical model probably, is one of the parametric model most extensively used in several fields; Lemonte et al. (2013). The popularity of this distribution can be explained perhaps, by the simplicity of their cumulative function, which involves only one unknown parameter $\lambda > 0$ and takes a simple form $G(x) = 1 - e^{-\lambda x}$ for $x > 0$ in addition to having constant hazard rate.

Gompertz (1825) and Verhulst [(1838), (1845) and (1847)] developed several cumulative distribution functions during the first half of the nineteenth century to compare known human mortality tables and represent mortality growth. One of them is as follows

$$G(t) = (1 - \rho e^{-\lambda t})^{\alpha} \quad (1)$$

for $t > {}^{1}/_{\lambda} In\rho$. Where ρ, λ and α are all positive real numbers. In twentieth century, Ahuja and Nash (1967) also considered this model and made some further generalization. The generalized exponential distribution or the exponentiated exponential distribution is defined as a particular case of the Gompertz (1825), Verhulst Verhulst [(1838), (1845) and (1847)] distribution function, when ρ = 1. Therefore, X is a two parameters generalized exponential random variable if it has the distribution function

$$G(x: \alpha, \lambda) = (1 - e^{-\lambda x})^{\alpha} \quad (2)$$

and the density function.

$$g(x: \alpha, \lambda) = \alpha\lambda(1 - e^{-\lambda x})^{\alpha-1} e^{-xt} \quad (3)$$

where α and λ play the role of the shape and scale parameters respectively. Many exponentiated families of distributions have appeared in the literature as generalizations of existing distributions. Mudholkar and Srivastava (1993) extended the Weibull distribution by introducing the 3-parameter exponentiated Weibull distribution (EWD) that has bathtub or monotone failure rate.

Gupta et al. (1998) studied the general properties of the exponentiated families of distributions such as hazard function and some ordering relations. Gupta and Kundu (1999) defined a 2-parameter generalized exponential distribution, a particular case of EWD, and studied some of its properties, including hazard rate, moment generating function, distribution of sums and extreme values. They also compared the flexibility of the generalized exponential distribution to a 2-parameter gamma distribution and a 2 parameter Weibull distribution by studying the deep groove ball bearings lifetime data. They concluded that the generalized exponential distribution can be used as alternative to the 2 parameter Weibull distribution and the 2-parameter gamma distribution.

Cadeiro et al. (2013) proposed a class of distributions by adding two parameters to a continuous distribution, by extending the idea first introduced by Lehman (1953) and studied by Nadarah and Kotz (2009). This method leads to a new class of Exponentiated generalized distribution (EG) that can be interpreted as a double construction of Lehmann alternative. The distributions extend the exponentiated type distribution and obtain some of its structure properties. Given a continuous c.d.f. G(x), we define the EG class of distributions by

$$F(x) = [1 - \{1 - G(x)\}^{\alpha}]^{\beta} \quad (4)$$

and

$$f(x) = \alpha\beta\{1 - G(x)\}^{\alpha-1}[1 - \{1 - G(x)\}^{\alpha}]^{\beta-1} g(x) \quad (5)$$

where $\alpha > 0 \; and \; \beta > 0$ are two additional shape parameters. The EG has tractable properties especially for simulation since its quantile function take a simple form.

$$x = Q_G\left(\left[1 - \left(1 - u^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}\right]\right) \quad (6)$$

where $Q_G(u)$ is the baseline quantile function.

To illustrate the flexibility of EG model, Cordero et al. (2013) applied EG to some well-known distribution such as the Frechet, normal, gamma and Gumbel distributions, with several properties for the EG class, which provide motivations to adopt this generator. The two extra parameters $\alpha \; and \; \beta$ in the density can control both tail weight, and allow generation of flexible distribution, with heavier or lighter tails, as appropriate. There is also an attractive physical interpretation of the EG model when $\alpha \; and \; \beta$ are positive integers see Cordeiro and Lemonte (2014). The EG family properties have been explored in recent works. Here, we refer to the papers: Cordeiro et al. (2014), Cordeiro and Lemonte (2014), Elbatal and Muhammed (2014), Oguntunde et al. (2014), da Silva et al. (2015), de Andrade et al. [(2016) and (2015)] Cordeiro et al. (2017), which used the EG class to extend the Burr III, Birnbaum-Saundersm, inverse Weibull, inverted exponential, generalized gamma, Gumbel, extended exponential, standardized half-logistic distributions respectively.

The rest of the paper is organized as follows. In Section 2 we define the Exponentiated Generalized Exponentiated Exponential (EGEE) distribution and outline some special cases of the distribution, the graphs of probability density function (pdf), cumulative distribution function (cdf) and hazard functions of proposed distribution and its sub-distributions are obtained. In section 3, some mathematical properties and limit behavior are derived, in section 4, estimation of the unknown parameters by method of maximum likelihood and information's criterion, in section 5, we provide some simulated result base on the mathematical properties and it real life application. We conclude in section 6 base on some significant result on the EGEE distribution.

II. METHODOLOGY

In this section we define and formulate the proposed model.

A Proposed Distribution (EGEE)

We defined the Exponentiated Generalized Exponentiated Exponential (EGEE) cumulative distribution from (4) as;

$$F(x) = [1 - \{1 - (1 - e^{-\frac{x}{\theta}})^{k}\}^{\alpha}]^{\beta} \quad (7)$$

By inserting (2) in (4) the corresponding p.d.f (5) is

$$f(x) = \frac{\alpha\beta k}{\theta} e^{-\frac{x}{\theta}}(1 - e^{-\frac{x}{\theta}})^{k-1}\{1 - (1 - e^{-\frac{x}{\theta}})^{k}\}^{\alpha-1}[1 - \{1 - (1 - e^{-\frac{x}{\theta}})^{k}\}^{\alpha}]^{\beta-1} \quad (8)$$

The hazard function is;

$$h(x) = \frac{\frac{\alpha\beta k}{\theta} e^{-\frac{x}{\theta}}(1 - e^{-\frac{x}{\theta}})^{k-1}\{1 - (1 - e^{-\frac{x}{\theta}})^{k}\}^{\alpha-1}[1 - \{1}{1 - [1 - \{1 - (1 - e^{-x/\theta})^{k}\}^{\alpha}} \quad (9)$$

The survival function is;

$$s(x) = 1 - [1 - \{1 - (1 - e^{-\frac{x}{\theta}})^{k}\}^{\alpha}]^{\beta} \quad (10)$$

B Special Case of EGEE Distribution

- The Exponential distribution (E) with scale parameter θ is a special case of EGEE when $\alpha = \beta = k = 1$.
- For $\alpha = \beta = 1$ the EGEE gives an Exponentiated Exponential (EE) distribution
- When $k = 1$ the EGEE gives a member of Exponentiated Generalized Family which is Exponentiated Generalized Exponential (EGE) distribution.
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Table 1 Summary of EGEE and Sub-Models

Distribution	β	α	k	θ
EGEE	β	α	k	θ
EGE	β	α	1	θ
EE	1	1	k	θ
E	1	1	1	θ

III. Mathematical Properties of The Proposed Distribution

We look at the some properties of EGEE model in this section.

A Properties of Exponentiated Generalized Exponentiated Exponential (EGEE)

The properties of the proposed distribution will be derived from Exponentiated Generalized Family (EGF) distribution in equation 1 and 2.

$$F(x) = \sum_{j=0}^{\infty} w_j\, G(x)^j \qquad (11)$$

where $w_j = w_j(\alpha, \beta) = \sum_{\lambda=0}^{\infty} \frac{(-1)^{\lambda+j}\Gamma(\beta+1)\Gamma(\alpha\lambda+1)}{\Gamma(\beta-\lambda+1)\Gamma(\alpha\lambda-j+1)\, j!\lambda!}$

Differentiating equation 12 with respect to x gives the pdf

$$f(x) = \sum_{j=0}^{\infty} jw_j g(x)\,(G(x))^{j-1} \qquad (12)$$

hence the cdf and pdf of the proposed distribution is express respectively as:

$$F(x) = \sum_{j=0}^{\infty} w_j\,((1 - e^{-\frac{x}{\theta}})^k)^j$$

and

$$f(x) = \frac{k}{\theta}\sum_{j=0}^{\infty} j\boldsymbol{w_j}\, e^{-\frac{x}{\theta}}(1-e^{-\frac{x}{\theta}})^{kj-1} \qquad (14)$$

B. Quantile and Median

$$x = -\theta[ln(1-\left(1-\left\{1-(p)^{\frac{1}{\beta}}\right\}^{\frac{1}{\alpha}}\right)^{\frac{1}{k}}\Big)\Big] \qquad (15)$$

$$Median = -\theta[ln(1 - \left(1-\left\{1-(0.5)^{\frac{1}{\beta}}\right\}^{\frac{1}{\alpha}}\right)^{\frac{1}{k}}\Big)\Big] \qquad (16)$$

C. Moment of EGEE Distribution

$$\mu^r = \int_0^{\infty} x^r f(x)\, dx \qquad (17)$$

$$\mu^r = \frac{k\theta^r}{(m+1)^{r+1}}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(kj)}{\Gamma(kj-m)\, m!} jw_j\ \Gamma(r+1) \qquad (18)$$

D. Order Statistics of EGEE

$$\frac{\propto \beta}{B(i, n-i+1)}\frac{k}{\theta} * \sum_{q=0}^{n-i}\sum_{p=0}^{\infty}\sum_{l=o}^{\infty}\sum_{d=0}^{\infty}(-1)^{p+q+l+d}\frac{\Gamma(n-i+1)\Gamma(\beta(q+i)}{\Gamma(n-i+q+\dots\ \Gamma(\alpha(p+1)-l)\Gamma(k(\dots} \qquad (19)$$

E. Skewness and Kurtosis of the EGEE Distribution

Galton (1883) proposed a quantile measure based approach for evaluating skewness while Moore (1988) did the same for Kurtosis. Galton's skewness and Moor's kurtosis is evaluated using the relations

$$S.K = \frac{Q(6/8) - 2Q(4/8) + Q(3/8) + Q(2/8)}{Q(6/8) - Q(2/8)} \qquad (20)$$

$$K.U = \frac{Q(7/8) - 2Q(5/8) + Q(3/8) + Q(1/8)}{Q(6/8) - Q(2/8)} \qquad (21)$$

Since the Quantile function of the EGEE distribution exists in closed form as given in (7), then (20) and (21) can be used in evaluating the skewness and kurtosis of the EGEE Distribution.

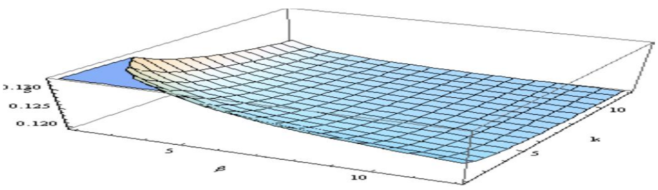

Figure 1 for EGEE Skewness

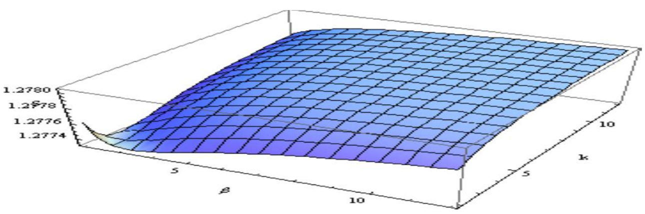

Figure 2 for EGEE Kurtosis

The 3D plot for skewness and kurtosis were plotted using the quartile function of the EGEE distribution with $\alpha = \theta = 1$ while $\beta = k$ takes values from 2 to 12.

F. Mean Deviation of EGEE Distribution

$$2\frac{k\theta}{(m+1)^2}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(kj)}{\Gamma(kj-m)m!}jw_j\,[1-\{1-(1-e^{-\mu/\theta})^k\}^\alpha]^\beta - 2\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(kj)}{\Gamma(kj-m)\,m!}jw_j\,\boldsymbol{\gamma}(2,\boldsymbol{\mu}) \quad \textbf{(22)}$$

G. Median Deviation of EGEE Distribution

$$\frac{k\theta}{(m+1)^2}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(kj)}{\Gamma(kj-m)m!}jw_j - 2\frac{k\theta}{(m+1)^2}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(kj)}{\Gamma(kj-m)\,m!}jw_j\,\boldsymbol{\gamma}(2,\mathbf{m}) \quad \textbf{(23)}$$

H. Asymptotic Behavior

We seek to investigate the behavior of the proposed model as given in Equation (8) as $x \to 0$ and as $x \to \infty$. This involves considering $\lim_{x\to 0} f(x)$ and $\lim_{x\to\infty} f(x)$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[\frac{\alpha\beta k}{\theta}e^{-\frac{x}{\theta}}(1-e^{-x/\theta})^{k-1}\{1-(1-e^{-x/})^k\}^{\alpha-1}[1-\{1-(1-e^{-\frac{x}{\theta}})^k\}^\alpha]^{\beta-1}\right] = 0$$

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty}\left[\frac{\alpha\beta k}{\theta}e^{-\frac{x}{\theta}}(1-e^{-x/\theta})^{k-1}\{1-(1-e^{-x/})^k\}^{\alpha-1}[1-\{1-(1-e^{-\frac{x}{\theta}})^k\}^\alpha]^{\beta-1}\right] = 0$$

These results confirm further that the proposed distribution has a mode (Oguntunde et al., 2014).

I. Maximum Likelihood Estimation

In this section, we determine the maximum likelihood estimates (MLEs) of the parameters of the EGEE distribution. For a random sample $x_1 x_2 \ldots . x_n$ of size n, the log-likelihood function of 4 parameter EGEE distribution is given by

$$L = \sum_{i=1}^{n}\ln(fx) = \sum_{i=1}^{n} ln\left(\frac{\alpha\beta k}{\theta}e^{-\frac{x}{\theta}}(1-e^{-\frac{x}{\theta}})^{k-1}\{1-(1-e^{-\frac{x}{\theta}})^k\}^{\alpha-1}[1-\{1-(1-e^{-\frac{x}{\theta}})^k\}^\alpha]^{\beta-1}\right)$$

$$nln\alpha + nln\beta + nlnk - nln\theta - \sum_{i=1}^{n}\frac{x_i}{\theta} + (k-1)\sum_{i=1}^{n}\ln\left(1-e^{-\frac{x_i}{\theta}}\right) + (\alpha-1)\sum_{i=1}^{n}\ln\{1-(1-e^{-\frac{x_i}{\theta}})^k\} + (\beta-1)\sum_{i=1}^{n}\ln[1-\{1-(1-e^{-\frac{x_i}{\theta}})^k\}^\alpha]$$

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln\{1-(1-e^{-\frac{x_i}{\theta}})^k\} + \frac{(\beta-1)\sum_{i=1}^{n}\ [1-\{1-(1-e^{-\frac{x_i}{\theta}})^k\}^\alpha]ln\{1-(1-e^{-\frac{x_i}{\theta}})^k\}}{[1-\{1-(1-e^{-\frac{x_i}{\theta}})^k\}^\alpha]}$$

$$\frac{\partial L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\ln[1-\{1-(1-e^{-\frac{x_i}{\theta}})^k\}^\alpha]$$

$$\frac{\partial L}{\partial k} = \frac{n}{k} + \sum_{i=1}^{n}\ln\left(1-e^{-\frac{x_i}{\theta}}\right) - (\alpha-1)\sum_{i=1}^{n}\frac{(1-e^{-\frac{x_i}{\theta}})^k ln\left(1-e^{-\frac{x_i}{\theta}}\right)}{\{1-(1-e^{-\frac{x_i}{\theta}})^k\}} + \alpha(\beta-1)\sum_{i=1}^{n}\frac{\{1-(1-e^{-\frac{x_i}{\theta}})^k\}^{\alpha-1}(1-e^{-\frac{x_i}{\theta}})^k ln\left(1-e^{-\frac{x_i}{\theta}}\right)}{[1-\{1-(1-e^{-\frac{x}{\theta}})^k\}^\alpha]}$$

$$\frac{\partial L}{\partial \theta} = \frac{-n}{\theta} + \sum_{i=1}^{n} \frac{x_i}{\theta^2} - (k+1) \sum_{i=1}^{n} \frac{x_i e^{-\frac{x_i}{\theta}}}{\theta^2 \left(1 - e^{-\frac{x}{\theta}}\right)} + (\alpha - 1) \sum_{i=1}^{n} \frac{k(1 - e^{-\frac{x_i}{\theta}})^{k-1} x_i e^{-\frac{x_i}{\theta}}}{\theta^2 \left(1 - \left\{1 - e^{-\frac{x}{\theta}}\right\}^k\right)}$$

$$-(\beta - 1) \sum_{i=1}^{n} \frac{\alpha k (1 - e^{-\frac{x_i}{\theta}})^{k-1} x_i e^{-\frac{x_i}{\theta}} \{1 - (1 - e^{-\frac{x_i}{\theta}})^k\}^{\alpha - 1}}{\theta^2 \left(1 \left(-1 - \left\{1 - e^{-\frac{x_i}{\theta}}\right\}^k\right)^\alpha\right)}$$

Solving the nonlinear system of equation of $\frac{\partial L}{\partial \alpha} = 0, \frac{\partial L}{\partial \beta} = 0, \frac{\partial L}{\partial k} = 0 \; and \; \frac{\partial L}{\partial \theta} = 0$ gives the maximum likelihood estimates of $\alpha, \beta, k \; and \; \theta$ respectively.

IV. Application

In this section, the proposed EGEE distribution is used to fit two (2) real-life data sets and estimates obtained were used to compare the fitted values of EGE, EE, and E distributions, respectively

Table 2: Maximum Likelihood Estimate of Parameters Standard Errors in Parenthesis, Loglikelihood and Information Criteria carbon Fibers

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{k}$	$\hat{\theta}$	*LL*	AIC	BIC
EGEE	11.3312	5.8340	3.2679	9.0031	-141.318	290.6361	301.0557
	(88.1057)	(45.3608)	(0.8815)	(15.6154)			
EGE	2.3577	7.7871		2.327	-146.182	298.3646	306.1801
	(102.0782)	(1.5042)		(100.7249)			
EE			7.79061	0.9868	-146.182	296.3646	301.575
			(1.4966)	(0.0851)			
E				0.3814	-196.371	394.7417	397.3469
				(0.0381)			

Source: Nichols and Padgett (2006)

Table 2 shows the reported values of log-likelihood function (LL) estimates for EGEE, EGE, EE and E distributions.

Table 3: Likelihood Ratio Test (LRT) Statistic for Carbon Fibers

Model	Hypothesis	LRT	$p - value$
EGEE vs. EGE	$H_0: \beta = 1 \; vs. H_1: H_0 \; is \; false$	9.7800	< 0.00001
EGEE vs. EE	$H_0: \beta = k = 1 \; vs. H_1: H_0 \; is \; false$	9.7800	< 0.00001
EGEE vs. E	$H_0: \beta = k = \alpha = 1 \; vs. H_1: H_0 \; is \; false$	110.1040	< 0.00001

Source: Nichols and Padgett, (2006)

Table 3 depicts the estimated (LRT) statistic for EGEE vs EGE, EGEE vs EE and EGEE vs E distributions.

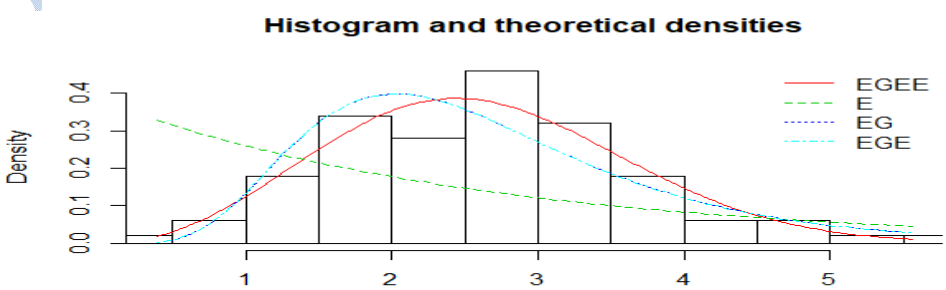

Figure 3: Fitted Plot of EGEE EGE, EE and E Models on Fibers Data

Figure 3 shows the histogram of the fibers data set along with fitted distributions.

Table 4: Maximum Likelihood Estimate of Parameters, Standard Errors in Parenthesis, Loglikelihood and Information Criterion Glass fibers data

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{k}$	$\hat{\theta}$	*LL*	AIC	BIC
EGEE	3.2594	45.663	7.710	2.188	-16.912	41.824	50.3971
	(2.705)	(37.900)	(1.6465)	(0.8228)			
EGE	64052.895	10.712		36621.219	-38.5517	83.1035	89.539
	(7278.081)	(3.1496)		(6444.711)			
EE			31.3627	0.3828	-31.3834	66.7669	71.0532
			(9.5330)	(0.0349)			
E				0.6636	-88.8303	179.6606	181.8038
				(0.0836)			

Source: Bourguinon et al. (2014).

Table 4 shows the reported values of log-likelihood function (LL) estimates for EGEE, EGE, EE and E distributions.

Table 5: Likelihood Ratio Test (LRT) Statistic for Glass Fibres

Model	Hypothesis	LRT	$p - value$
EGEE vs. EGE	$H_0: \beta = 1 \; vs. H_1: H_0 \; is \; false$	43.2794	< 0.00001
EGEE vs. EE	$H_0: \beta = k = 1 \; vs. H_1: H_0 \; is \; false$	28.9428	< 0.00001
EGEE vs. E	$H_0: \beta = k = \alpha = 1 \; vs. H_1: H_0 \; is \; false$	143.8366	< 0.00001

Source: Bourguinon et al. (2014).

Table 5 depicts the estimated (LRT) statistic for EGEE vs. EGE, EGEE vs. EE and EGEE vs. E distributions.

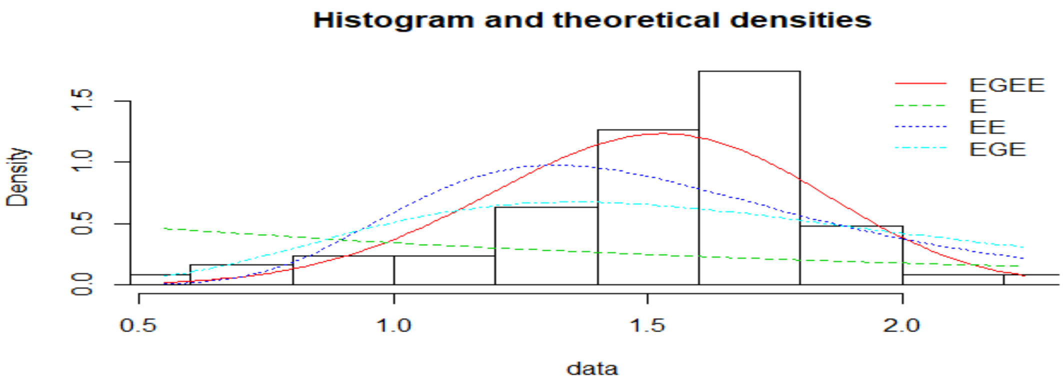

Figure 4: Fitted Plot of EGEE EGE, EE and E Models on Glass Fibres Data

Figure 4 shows the histogram of the glass fibers data set along with fitted distributions

V. Discussions

Table 2 shows the reported values of log-likelihood function (LL) estimates for EGEE, EGE, EE and E distributions obtained as -141.318, -146.182, -146.182 and -196.371 respectively with Akaike Information Criterion (AIC) values of 290.6361, 298.3646, 296. 3646, and 394. 7417. The Bayesian Information Criterion (BIC) indicates the values of 301.0557, 306.801, 301.575, and 397.3469, respectively, for all the distributions, as mentioned above. The estimated result confirmed that the proposed distribution EGEE with values of -141.318, 290.6361 and 301.0557 for LL, AIC, and BIC respectively gives a better distribution fit for the data set. The EGE and EE distribution

performed equally on the data set as the likelihood values confirm it in Table 2.
Table 3 depicts the estimated (LRT) statistic for EGEE vs EGE, EGEE vs EE and EGEE vs E distributions of reported values 9.7800, 9.7800 and 110.1040 respectively and the corresponding p-values of 0.001, 0.001 and 0.00001. The finding reveals that **EGEE** distribution is significant at 0.05 level of significance to EGE, EE, and E distributions. Figure 3 shows the histogram of the fibers data set along with fitted distributions. The theoretical density of EGEE distribution depicts the best spread compared to EGE, EE, and E distributions on the data set.
Table 4 shows the reported values of log-likelihood function (LL) estimates for EGEE, EGE, EE and E distributions are obtained as -16.912, -38.5517, 31.3834and -88.8303 respectively with Akaike Information Criterion (AIC) values of 41.824, 83.1035, 66.7669 and 179.6606. The Bayesian Information Criterion (BIC) indicates the values of 50.3971, 89.539, 71.0532, and 181.8038, respectively, for all the distributions, as mentioned earlier. The estimated result confirmed that the proposed distribution EGEE with values of -16.912, 41.824 and 50.3971 for LL, AIC, and BIC respectively and therefore gives the best fit for the data set.

Table 5 depicts the estimated (LRT) statistic for EGEE vs. EGE, EGEE vs. EE and EGEE vs. E distributions of reported values 43.2794, 28.9428 and 143.8366respectively and the corresponding p-values of 0.0001, 0.0001 and 0.000001. The finding reveals that EGEE distribution is significant at 0.05 level of significance to EGE, EE, and E distributions.

Figure 4 shows the histogram of the glass fibers data set along with fitted distributions. The theoretical density of EGEE distribution has a better spread than its sub-distributions on the data set.

VI. Conclusion

The application of the EGEE and its sub-distributions on Fibres and glass data in Tables 3 and 4 shows that the proposed model serves as a best distribution compared to its sub-distributions using the information criterion (AIC and BIC), log likelihood, likelihood ratio test and the fit distribution curve, while the data on precipitation in Table 4.50 and its fitted plot on Figure 4.23, show that EGEE distribution can serve as an alternative and better distribution where the sub-distributions are applied.

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