On Statistical Properties of the Weibull Inverted Weibull Distribution

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***Abstract* — A new distribution called Weibull Inverted Weibull Distribution (WIWD) was introduced. The new distribution was used in analysing life-times data that exhibits bathtub failure rates. We considered the standard Weibull Inverted Weibull Distribution (WIWD) that generalizes the standard Inverted Weibull Distribution (IWD), the new distribution has two shape parameters which contributes to its flexibility. The moments, mean, variance, reliability function, hazard function, moment generating function, and the order statistics were obtained. A real data set was analysed and it was observed that the (WIWD) distribution can provide a better fitting than (IWD) distribution.**

Keywords- ***Moments, Bathtub failure rate, Inverted Weibull distribution*.**

I. INTRODUCTION

The inverted Weibull distribution is one of the most popular probability distribution to analyze the life time data with some monotone failure rates. Ref. [5] explained the flexibility of the three parameters inverted Weibull distribution and its interested properties. Ref. [1] studied the properties of the inverted Weibull distribution and its application to failure data. Ref. [2] introduced the exponentiated Weibull distribution as generalization of the standard Weibull distribution and applied the new distribution as a suitable model to the bus-motor failure time data. Ref. [3] reviewed the exponentiated Weibull distribution with new measures. In this article we use the generalization of the Weibull –G family of distribution which was introduced by [4] with the density function is given by

$$F(x) = 1 - exp\left\{\eta\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{\alpha}\right\}, \quad x > 0;\ \eta, \alpha > 0 \quad (1)$$

Where $G(x;\xi)$ is the baseline cdf, which depends on a parameter vector ξ. By differentiating equation (1), we obtain the probability density function given as

$$f(x) = \alpha\eta g(x;\xi)\frac{G(x;\xi)^{\alpha-1}}{\bar{G}(x;\xi)^{\alpha+1}} exp\left\{\eta\left[\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right]^{\alpha}\right\} \quad (2)$$

It should be noted that, $\bar{G}(x; \xi = 1 - G(x;\xi)$.

A brief explanation on the Weibull-G family can be found in (Cooray, 2006). Let Y be a lifetime random variable having a continuous G distribution, then the odd ratio that an individual (or component) following Y will die or fail at time t is given as $\frac{G(x;\xi)}{\bar{G}(x;\xi)}$. Suppose that the variability of this odds of death is represented by the random variable X and also assumes that it follows the Weibull model with the scale parameter α and shape parameter η. Then we can write

$$\Pr(Y \le x) = Pr\left(X \le \frac{G(x;\xi)}{\bar{G}(x;\xi)}\right) = F(x;\alpha,\eta,\xi) \quad (3)$$

Which can be obtained using the expression in equation (2)

II. WEIBULL INVERTED WEIBULL DISTRIBUTION (TIWD)

Random variable X has a standard inverted Weibull distribution (IWD) if its distribution function takes the following form:

$$G(x;\beta) = e^{-x^{-\beta}} \qquad x > 0, \quad \beta > 0 \quad (4)$$

The pdf of Inverted Weibull distribution is given by

$$g(x;\beta) = \beta x^{-\beta} e^{-x^{-\beta}} \quad (5)$$

Now using (4) and (5) in (1) we have the cdf of a (WIWD) given by

$$F(x) = 1 - exp\left\{\eta\left[\frac{e^{-x^{-\beta}}}{1 - e^{-x^{-\beta}}}\right]^{\alpha}\right\}, \quad x > 0;\ \eta, \alpha, \beta > 0 \quad (6)$$

The graph of the cdf of WIWD is given by Fig 1.

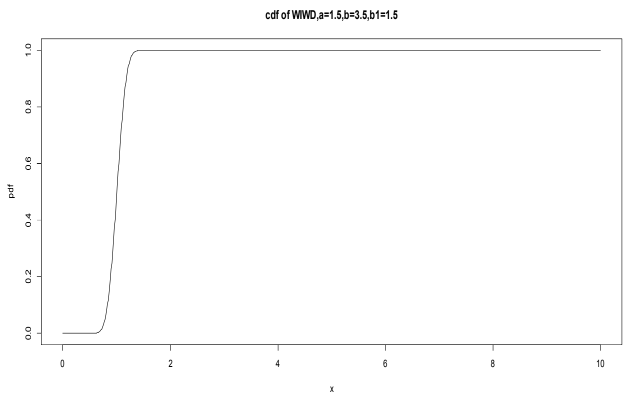

Fig. 1.0 The graph of the cumulative density function of WIWD.

The figure (Fig 1) indicates that the cdf of WIWD is a proper or true cdf.

The probability density function of the WIWD is given as

$$f(x) = \alpha\eta\beta x^{-\beta} e^{-x^{-\beta}} \frac{(e^{-x^{-\beta}})^{\alpha-1}}{(1-e^{-x^{-\beta}})^{\alpha+1}} e^{\left\{\eta\left[\frac{e^{-x^{-\beta}}}{1-e^{-x^{-\beta}}}\right]^{\alpha}\right\}} \qquad (7)$$

When we consider, $e^{\left\{\eta\left[\frac{e^{-x^{-\beta}}}{1-e^{-x^{-\beta}}}\right]^{\alpha}\right\}}$, and applying the power series for the exponential function, we obtain

$$e^{\left\{\eta\left[\frac{e^{-x^{-\beta}}}{1-e^{-x^{-\beta}}}\right]^{\alpha}\right\}} = \sum_{k=0}^{\infty} \frac{(-1)^k \eta^k}{k!} \left[\frac{e^{-x^{-\beta}}}{1-e^{-x^{-\beta}}}\right]^{k\alpha} \qquad (8)$$

Inserting this in equation (7) we have,

$$f(x) = \alpha\eta\beta x^{-\beta} e^{-x^{-\beta}} \frac{(e^{-x^{-\beta}})^{\alpha-1}}{(1-e^{-x^{-\beta}})^{\alpha+1}} \sum_{k=0}^{\infty} \frac{(-1)^k \eta^k}{k!} \qquad (9)$$

Finally the pdf of WIWD can be represented as

$$f(x;\alpha,\eta,) = W_{ik}\alpha\eta\beta x^{-\beta} e^{-(\alpha(k+1)+i)x^{-\beta}} \qquad (10)$$

where

$$W_{ik} = \sum_{k=0}^{\infty}\sum_{i=0}^{\infty} \frac{\Gamma(\alpha(k+1)+i+1)}{i!\,\Gamma(\alpha(k+1)+1)} \frac{(-1)^k \eta^k}{k!} \qquad (11)$$

The graph of the pdf for various values of the parameters is given by Fig 2.

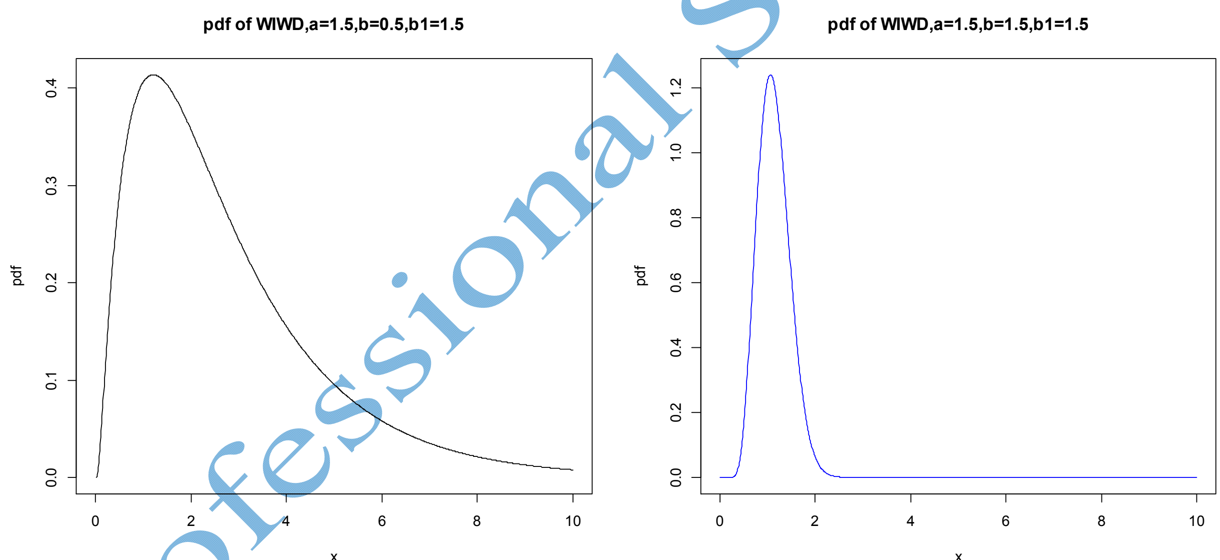

Fig. 2 The graph of the pdf of WIWD for various values of the parameters.

III. MOMENTS, MEAN, VARIANCE OF WIWD

In this section we shall obtain the moments (WIWD). The r^{th} order moments, of (WIWD) can be obtained as follows for a random variable *X*,

$$E(X)^r = \int_{-\infty}^{\infty} x^r f(x)dx \qquad (12)$$

Putting eq. (10) in eq. (12), we have

$$E(X)^r = W_{ik}\alpha\eta\beta \int_{-\infty}^{\infty} x^{r-\beta} e^{-(\alpha(k+1)+i)x^{-\beta}} \quad (13)$$

If we let $p = (\alpha(k+1)+i)x^{-\beta}$ in eq.(13) then we have

$$E(X)^r = \mu_r = \frac{-W_{ik}\alpha\eta}{\{\alpha(k+1)+i)\}^{2-\frac{r}{\beta}}}\Gamma\left\{\frac{\beta-1-r}{\beta},\left(\frac{t}{\alpha(k+1)+i)}\right)^{-\frac{1}{\beta}}\right\} \quad (14)$$

The first and second moment and also an expression of the variance is given as follows:

$$\mu_1 = \frac{-W_{ik}\alpha\eta}{\{\alpha(k+1)+i)\}^{2-\frac{1}{\beta}}}\Gamma\left\{\frac{\beta-2}{\beta},\left(\frac{t}{\alpha(k+1)+i)}\right)^{-\frac{1}{\beta}}\right\} \quad (15)$$

$$\mu_2 = \frac{-W_{ik}\alpha\eta}{\{\alpha(k+1)+i)\}^{2-\frac{2}{\beta}}}\Gamma\left\{\frac{\beta-3}{\beta},\left(\frac{t}{\alpha(k+1)+i)}\right)^{-\frac{1}{\beta}}\right\} \quad (16)$$

The mean of (WIWD) is the first moment about the origin (μ_1)which corresponds to eq. (15)

And the variance of (WIWD) can be obtained using the relation

$$V(X) = \mu_2 - (\mu_1)^2 \quad (18)$$

Inserting eq. (15) and eq. (16) in eq. (17) we have

$$V(X) = \frac{-W_{ik}\alpha\eta}{\{\alpha(k+1)+i)\}^{2-\frac{2}{\beta}}}\Gamma\left\{\frac{\beta-3}{\beta},\left(\frac{t}{\alpha(k+1)+i)}\right)^{-\frac{1}{\beta}}\right\} - \left[\frac{-W_{ik}\alpha\eta}{\{\alpha(k+1)+i)\}^{2-\frac{1}{\beta}}}\Gamma\left\{\frac{\beta-2}{\beta},\left(\frac{t}{\alpha(k+1)+i)}\right)^{-\frac{1}{\beta}}\right\}\right]^2 \quad (19)$$

IV. Reliability Analysis

The reliability function define as, $R(x) = 1 - F(x)$, for (WIWD) is given by:

$$R(x) = e^{\left\{\eta\left[\frac{e^{-x^{-\beta}}}{1-e^{-x^{-\beta}}}\right]^{\alpha}\right\}} \quad (20)$$

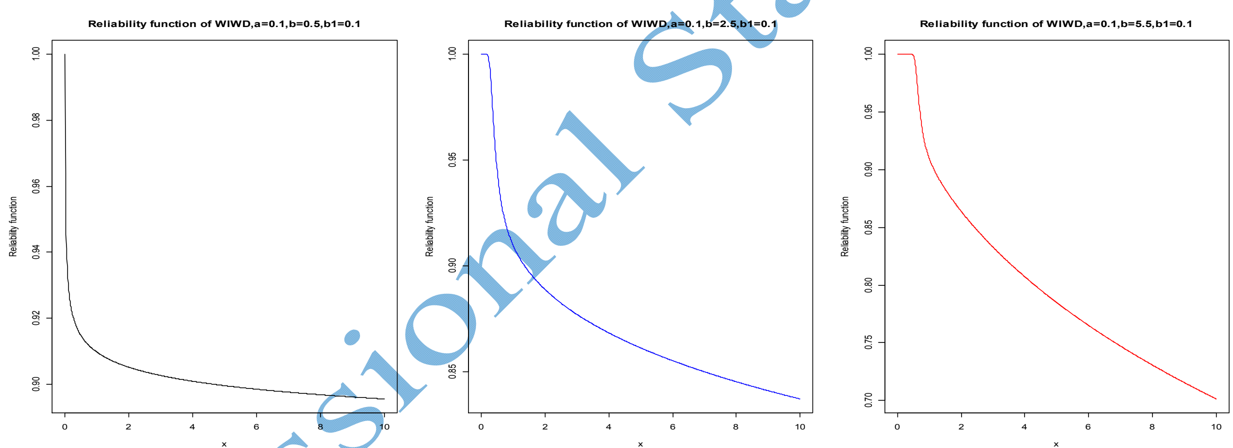

Fig. 4: The graph of the reliability function for various values of the parameters

V. Hazard Function

The hazard rate function also known as instantaneous failure rate defined by

$$h(x) = \frac{f(x)}{1-F(x)} \quad (21)$$

$$h(x) = \alpha\eta\beta x^{-\beta}(e^{-x^{-\beta}})^{\alpha}(1-e^{-x^{-\beta}})^{-(\alpha+1)} \quad (22)$$

The graphs of the hazard function for various values of the parameters is given Fig 5 and 6.

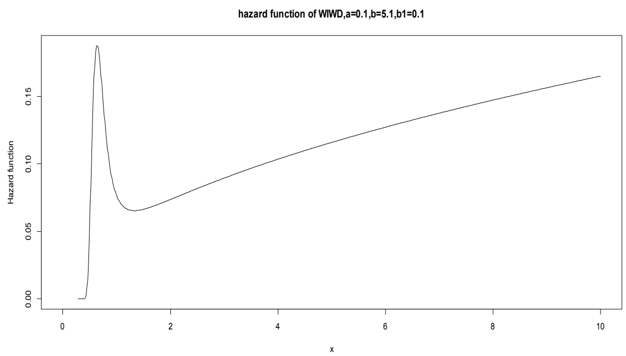

Fig. 5: The graph of the hazard function of WIWD (the hrf is increasing, decreasing and increasing.

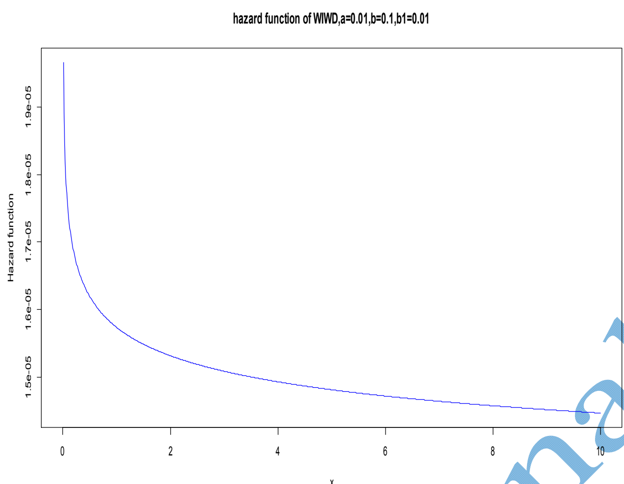

Fig.6: The graph of the hazard function of WIWD (the hrf is strictly decreasing)

VI. Moment generating function of (WIWD)

The moment generating function of a random variable x is defined by

$$M_t(x) = \int_{-\infty}^{\infty} e^{tx} f(x)dx \tag{27}$$

The above expression can further be simplify as

$$M_t(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_{-\infty}^{\infty} x^k f(x)dx \tag{28}$$

Since,

$$e^{tx} = \sum_{r=0}^{\infty} \frac{t^k x^k}{k!} \tag{29}$$

Inserting eq. (15) in eq. (28) we have

$$M_t(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\left(1 - \frac{k}{\beta}\right)\left\{\lambda\left[\left(\frac{1}{2}\right)^{-\frac{k}{\beta}} - 1\right] - 1\right\} \tag{30}$$

The above expression is the moment generating function of (WIWD)

VII. Applications

In order to show the capability of the WIW distribution we apply the distribution to a failure data which exhibits non-monotone failure rate. The following criteria were used as a measure of it its fit compared to its sub-model, that it's the Smirnov (K-S) statistics, Anderson Darling statistic (A^*), crammer Von Misses (V), Akaike information criterion (AIC), Consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQIC). The real data set represents the remission times (in months) of a random sample of 128 bladder cancer patients [7]. This dataset is known to have a unimodal hazard rate function shaped in the literature.

The Exploratory data analysis is given in table 1. The results of this application to the real data set are listed in Table 2 and Table 3. These results indicates that the WIW distribution has a lower AIC, CAIC, BIC, HQIC, K-S, A^*, and W values and has the biggest estimated log-likelihood (l) and p-value of the K-S statistics among all the fitted models. Hence, it could be chosen as the best model under these criteria.

Table 1.0 Descriptive Statistics on Cancer remission time.

Min	Q_1	Median	$mean$	Q_3	Max	$kurtosis$	Skewness	$Variance$
0.0047	0.3723	0.4734	0.4810	0.6405	0.9774	16.1537	3.3256	110.425

Table 2.0 Estimated parameters of the WIWD and IWD

$Model$	$Estimates$			$l(\hat{\theta})$
$WIWD$ (α,η,β)	3.0076 (0.01158)	-2.3164 (0.7408)	-0.8889 (0.2492)	-754.592
IWD (β)	- $(-)$		0.6369 (0.01157)	-953.623

Table 3 Measures of Goodness of Fit

$\boldsymbol{Model}$	$\boldsymbol{K-S}$	A^*	$\boldsymbol{W}$	$\boldsymbol{AIC}$	$\boldsymbol{BIC}$	HQIC	CAIC
$\boldsymbol{TIWD}$	0.0232	0.6309	0.07028	1515.185	1529.908	1520.781	1515.209
$\boldsymbol{IWD}$	0.6412	42.9677	7.3879	1909.247	1914.154	1911.112	1909.251

VIII. Conclusion

Among the models considered the WIW distribution may be considered to be a better model than the inverted Weibull distribution for the data set.

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