Enhanced Hotelling T² Technique Using Quartile Mean

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Abstract — **This research, focused on enhancing the performance of the traditional Hotelling T^2 technique using quartile mean in the presence of outliers. An alternative approach to Hotelling T^2 technique based on quartile mean was proposed. The performance of the proposed technique was assessed in comparison to the traditional Hotelling T^2 technique and the existing Hotelling T^2 (decile) and Hotelling T^2(trim) techniques based on probability of detection. The credibility of the approach was investigated through a real data set on the estimated use of water in the United States in the year 2010. Results revealed that the efficiency of traditional Hotelling T^2 technique is influenced when outliers are present in a data set resulting in the reduction of it's probability of detection. Hotelling T^2 (decile) equally exhibited low efficiency in outlier detection with Hotelling T^2(trim) technique having a reasonably high probability of detection. The proposed alternative to Hotelling T^2 technique; Hotelling T^2(quartile) out-performed all the other techniques considered giving the highest probability of detection which indicates a better performance in detecting abnormalities in the data set. Therefore this study recommends the use of Hotelling T^2 (quartile) technique for the computation of Hotelling T^2 statistic when outliers are present in the data set and when the data is not normally distributed.**

Keywords- Hotelling T^2 technique; trimmed mean; decile mean; quartile mean; probability of detection.

I. INTRODUCTION

The first original study in multivariate quality control was introduced by Hotelling (1947) (Hotelling, 1947). Prior to this study, he wrote a paper on T^2 - test procedures for multivariate population in 1931 (Hotelling, 1931). Ever since, the Hotelling T^2 technique found widespread use during World War II and has been employed with various modifications (Jackson, 1985). Multivariate process control has gained acceptance in statistical process control and the quest to improve the MSPC in a production process is on high demand Lowry and Montgomery (2007). Following that, many authors have conducted studies on Hotelling T^2 control chart. Amongst them are; Alfaro and Ortega (2008), (Abuwiesh *et al*., 2012), (Hanif *et al* 2013) and (Ali *et al*., 2013).

In many industrial processes, there are many situations in which the simultaneous monitoring or control of two or more quality characteristics is necessary Umit and Cigdem (2001). Over time, multivariate quality control charts have been employed to monitor many process variables simultaneously thereby giving it an edge over simultaneous use of univariate chart (Akeem *et al.,* 2015).

In order for companies and industries to have a healthy competition in the market, the quality of their products and process of production is of great priority. Data errors increase in manufacturing processes due to huge collection of data. These errors may be due to the presence of outliers or departure from the normal distribution (Brooks, 1985).

An outlier is defined as an observation that deviates so much from other observations as to arouse suspicion that it was generated by a different mechanism (Hawkins, 1980). The assumption of normality in the use of statistical quality control tools especially in real life data set is rarely true (Hanif *et al.*, 2013). The presence of outliers will affect the accuracy of the Hotelling T^2 technique in detecting abnormalities (Hanif *et al.*, 2013). It leads to reduction of the probability of detection in Phase I, which consequently reduces the power to detect changes in Phase II process of the classical Hotelling T^2 technique (Ali *et al*., 2013).

A classical procedure can be shown to be optimal only under a series of assumptions such as Normality, linearity, symmetry, independence or finite moments (He, 1990). Violations of these distributional assumptions often nullifies the optimality seriously, even more dangerous is the occurrence of outliers (He, 1990). For example, with just one bad point, the sample mean which is one of the parameters in the computation of the Hotelling T^2 statistic can go everywhere, yielding no relevant information at all (He, 1990). The presence of multiple outliers may go undetected by the usual Hotelling T^2 technique due to the masking effect (Alfaro *et al*., 2008).

II. Multivariate Statistical Process Control

Multivariate statistical process control is the term used to describe process monitoring problems in which several related variables are of interest Umit and Cigdem (2001). Umar *et. al.,* (2017) carried out a multivariate study on the physicochemical quality of chloroquine tablet using classical Hotelling T^2 control chart and MEWMA control chart. They concluded that the MEWMA control chart is observed to be more sensitive, effective and reliable than the Hotelling T^2 control chart in detecting shifts in a process. They recommended the need to explore the robust Hotelling T^2 control chart as an alternative to both the classical MEWMA and classical Hotelling T^2 control chart for better detection of shifts in product process means.

The implementation of robust technique in classical Hotelling T^2 control chart helps to overcome the effect of outliers in phase one and phase two of quality control process (Hanif *et al*., 2013). A popular strategy is used to make multivariate approaches more efficient by replacing the location and the scale estimators with measures of central tendency and dispersion that are resistant to outliers Shabbak and Midi (2012). It is in this same view that (Abu-Shawiesh *et al*., 2012) focused on a new bivariate control chart for *m* sub-groups based on the robust estimators as an alternative to the traditional Hotelling's T^2 control chart.

The location vector and the variance-covariance matrix for the new technique are obtained using the sample median, the median absolute deviation from the sample median, and the comedian estimator (the covariance between two random variables). The performance of the proposed methods in detecting outliers was evaluated and compared with the Hotelling's T^2 method using a Monte-Carlo simulation study. They concluded that, the performance of the proposed method is better, and has a superior behavior over the non-robust one. They also added that the approach can work for elliptically countered type of bivariate data but most likely, it will encounter difficulties if a higher dimensional data is used. Also in a bid to improve the Hotelling T^2 control chart (Sindhumol *et al.,* (2016) studied a robust dispersion control chart based on modified trimmed standard deviation.

This study focused on variability due to dispersion of a quality Characteristic. They introduced a modification to trimmed standard deviation to increase it's efficiency. The proposed robust control chart was compared with s-chart and they concluded that the robust control chart performed remarkably.

Alfaro *et al*., (2008) studied a robust alternative to Hotelling's T^2 control chart using trimmed estimators. They replaced the sample mean vector in the traditional Hotelling's T^2 statistic by the trimmed mean vector, and the variance covariance matrix by the trimmed variance covariance matrix to construct the robust alternative to Hotelling T^2 statistic. They concluded that the new robust Hotelling T^2 statistic is more effective in detection of outliers. (Hanif *et al.,* 2013) introduced two robust means for the computation of the Hotelling T^2 statistic. Decile and trimmed means were used to replace the ordinary mean in phase one. The performance of their proposed approaches was assessed based on probability of detection using a simulation study and a real data set. Results revealed that the proposed alternatives outperformed classical Hotelling T^2 technique in outlier detection.

Also, Haddad *et al*. (2013) considered improving the performance of the traditional Hotelling T^2 control chart. In this study, the usual mean vector in the Hotelling T^2 chart was replaced by the winsorized modified one-step M-estimator (MOM) whereas the usual covariance matrix was replaced by the winsorized covariance matrix. MOM empirically trims the data based on the shape of the data distribution. The performance of each control chart is based on the false alarm and the probability of outlier's detection. They concluded that the performance of the alternative robust Hotelling's T^2 control chart is better than the performance of the traditional Hotelling's T^2 control chart.

Franklin *et al*.(2001) illustrated the practical application of a robust multivariate outlier detection method used to edit survey data in which outliers were identified by calculating Mahalanobis distance, where the location vector and scatter matrix were robustly estimated using modified Stahel-Donoho estimators.

Iglewicz and Langenberg (2018) in their study introduced a modified approach to the computation of control limits for $\bar{X}$ and R charts. This procedure consists of replacing $\bar{X}$ with the trimmed mean of the subgroup averages, and $\bar{R}$ with the trimmed mean of the subgroup ranges. The proposed control chart limits are shown to be less influenced by extreme observations than their classical counterparts, and to lead to tighter limits in the presence of out-of-control observations.

Sullivan and Woodall (1996) carried out a comparative study of individual observations. Several alternatives for estimating the covariance matrix were compared and a procedure analogous to the use of moving ranges in the univariate case was recommended. This procedure uses the vector difference between successive observations to estimate the in-control covariance matrix of the process. The result from this study shows that the Hotelling T^2 technique is not effective in detecting small shifts in mean vector which reduces the alarm signals.

Furthermore, Alfaro and Ortega (2009) developed four alternatives to the traditional Hotelling's T^2 control chart. These proposed control charts used minimum volume ellipsoid (MVE) estimator, minimum covariance determinant (MCD) estimator, reweighted (MCD) estimator and the trimmed mean estimator. They concluded that the robust alternatives to Hotelling T^2 control chart behaved better than the traditional Hotelling T^2 control chart in the presence of outliers and further recommended the use of Robust Hotelling T^2 charts that depend on the trimmed mean and the modified of the MCD estimators when the amount of outliers is small.

Vargas (2003) introduced robust control charts for identifying outliers in Phase I of multivariate individual observations based on two robust estimates of mean vector and covariance matrix, namely, the minimum covariance determinant (MCD) and the minimum volume ellipsoid (MVE). The performance of the robust control charts was assessed based on the probability of signal in Phase I only. It was concluded that the robust alternatives outperformed the traditional Hotelling T^2 control chart.

Ali *et al.* (2013) in their study, developed an alternative robust control chart based on minimum vector variance (MVV) estimator. Results showed that MVV was able to detect out-of-control signal and simultaneously control false alarm rate even as the dimension increased. It was also observed that the MVV maintained its good performance in terms of false alarm and probability of detection.

Mahammadi and Arasan (2010) and Chenouri and Variyath (2009) constructed a new statistic by substituting the classical estimators in Hotelling's T^2 by the MCD and reweighted MCD (T^2 RMCD). Their simulation studies showed that, when there are outliers in phase I, the T^2 R MCD is more effective than the standard T^2 and the ordinary MCD charts.

Yahaya *et al.* (2011), in their study proposed a robust Hotelling T^2 control chart for individual observations based on minimum vector variance (MVV) estimators. Results showed that MVV control chart has competitive performance relative to MCD and traditional control charts even under certain location parameter shifts in Phase I data. These studies clearly illustrated the superiority of the robust Hotelling T^2 control chart approaches over the classical control chart in terms of outlier detection, probability of detection and false alarm rates even when dimension increases.

Abu-Shawiesh *et al*. (2014) studied a robust bivariate control chart alternative to the Hotelling's T^2 control chart. They considered the following robust alternatives to the classical Hoteling's T^2, T^2MedMAD, T^2MCD, T^2MVE. A simulation study was conducted to compare the performance of these control charts. Two real life data were analyzed to illustrate the application of these robust alternatives to the proposed robust method. It was observed that T^2MedMAD has the lowest false alarm rate while having the highest power.

Mostajeran *et al.,* 2016) investigated a New Bootstrap Based Algorithm for Hotelling's T^2 Multivariate Control Chart. The performance of the proposed chart was evaluated through a simulation study and a real data set based on ARL_0 and ARL_1 assuming multivariate Normal, multivariate *t*, multivariate skew-Normal and multivariate lognormal distributions. The results of this study was compared to the traditional Hotelling's T^2 technique and the bootstrap results reported by Phaladiganon *et al.* (2011). It was concluded that the proposed algorithm performed better than the above mentioned methods.

Williams *et al.* (2018) studied the distribution of Hotelling T^2 statistic based on the successive differences covariance matrix estimator. This study demonstrated several useful properties of the T^2 statistics based on the successive differences estimator and gave a more accurate approximate distribution for calculating the upper control limit for individual observations in a Phase I analysis. The Hotelling T^2 chart performance with the proposed control limit, which varied with the position of the observation, was studied. It was concluded that, with the proposed limit the actual false alarm probability is much closer to the specified value with small samples.

Despite so many proposed alternatives to Hotelling T^2, researchers in this field have not relented in exploring newer methods of abating the above mentioned challenges. In a continuous quest to improve the performance of Hotelling T^2 technique, this study reviewed the robust approaches to Hotelling T^2 technique proposed by (Hanif *et al*., 2013) based on robust means; Hotelling T^2 (trim) and Hotelling T^2 (decile) and proposed an alternative method to traditional Hotelling T^2 technique; Hotelling T^2 (quartile). The proposed Hotelling T^2 (quartile) is studied in comparison to the traditional Hotelling T^2 and the existing methods based on higher probability of detection using a real data set in the presence of outliers.

III. Multivariate Hotelling T^2

The Hotelling T^2 technique is a tool to detect multivariate outliers, mean shifts, and other distributional deviations from the in-control distribution (Williams *et al*., 2018). In any multivariate statistical process control application, generally two phases are considered (Alt, 1985). Suppose that there is a historical data set (HDS) in the phase I monitoring scheme that consists of n observation vectors of dimension p, which are observed independently, where p is the number of dimensions that are measured ($p < n$). It is assumed that each vector comes from a p-variate normal distribution. Thus, if $x_i \epsilon \mathcal{R}^p$ is a vector in the HDS for the i^{th} time period, $X_i \sim N_p(\mu, \Sigma)$ where μ and Σ are the

population mean vector and the variance-covariance matrix respectively Shabbak and Midi (2012).
The general form of this statistic is

$$T^2 = (X_i - \mu)\Sigma^{-1}(X_i - \mu)' \qquad (1)$$

According to (Williams *et al*., 2018), the parameters in (1) are usually unknown and the usual sample mean and variance-covariance matrix are used as the classical estimations of μ and Σ. In practice, these variables are expressed by

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad (2)$$

$$S^2 = \frac{1}{n-1}\sum_{I=1}^{n}(X_i - \bar{X})^2 \qquad (3)$$

The covariance between, say X_i, X_p is generally given by

$$S_{ip} = Cov(X_i, X_p) \qquad (4)$$

$$= \mathrm{E}\big[(X_i - \mu_i)(X_p - \mu_p)\big] \qquad (5)$$

$$= \mathrm{E}\big(X_i X_p\big) - \mu_i\mu_p \qquad (6)$$

Clearly, when $p = i$ we obtain the variance

$$S_{ip} = \mathrm{E}[(X_i - \mu_i)^2] \qquad (7)$$

For *p*- variables, set the covariance matrix as

$$\Sigma = S_{ip} = \begin{bmatrix} S_{1,1} & S_{1,2} & S_{1,3} & \cdots & S_{1p} \\ S_{2,1} & S_{2,2} & S_{2,3} & \cdots & S_{2,p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_{i,1} & S_{i,2} & S_{i,3} & \cdots & S_{i,p} \end{bmatrix} \qquad (8)$$

The phase I of the monitoring scheme consists of collecting a sufficient number of data to ascertain whether or not the historical data indicate a stable (or in-control) process. The phase I analysis is sometimes called retrospective analysis (Shabbak *et al*., 2011).
The phase I UCL is given by;

$$UCL \sim \chi^2_{(1-\alpha),p} \qquad (9)$$

Whereas in phase II, future observations are monitored based on the control limits calculated from Phase I to determine if the process continues to be a stable process or not. In phase II, the control limit is as follows:

$$UCL \sim \left[\frac{P(n+1)(n-1)}{n(n-P)}\right], F_{1-\alpha}(P, n-P) \qquad (10)$$

The use of (1) is not effective in the presence of multiple outliers, so an alternative method is proposed. In this regard, this study proposed an alternative approach to Hotelling T^2 based on quartile mean, which will be denoted by Hotelling $T^2_{(Quartile)}$ and is defined as follows:

$$T^2{}_{(Quartile)i} = (X_i - \bar{X}_{(Quartile)})\,\Sigma^{-1}\big(X_i - \bar{X}_{(Quartile)}\big)' \qquad (11)$$

Where $\bar{X}_{(Quartile)}$ is the alternative estimator of the sample mean vector and Σ^{-1} is the inverse of the sample variance-covariance matrix.

a. Quartile Mean

Quartiles are the values dividing the whole observations into 4 equal parts (Bajracharya *et al*., 2017). The quartile is given by;

$$Q_i = \text{Size of } \left[\frac{i(N+1)}{4}\right]^{th} \text{ item of the series for } i = 1,2,3 \qquad (12)$$

Therefore, the quartile mean (QM) here is computed as the summation of the individual quartiles divided by the total number of quartiles i.e.

$$QM = \frac{Q_1 + Q_2 + \cdots + Q_3}{3} \qquad (13)$$

where *QM* is the quartile mean.

b. Hotelling T^2 Technique with Quartile Mean in Phase I

To enhance the performance of the classical Hotelling T^2 technique, quartile mean $\bar{X}_{(Quartile)}$ will be used in phase I to replace ordinary mean $\bar{X}$
The following procedures explain the details:

Phase I

Step 1 : Using sample size n, dimensions *p*, and confidence level $(1 - \alpha)$
Step 2 : Collect the Phase I data $(x_1, x_2, \ldots, x_n)$ at well-defined periodic intervals.
Step 3 : Use the Phase I data, to compute quartile mean $\bar{x}_{(Quartile)}$ and covariance Σ^{-1}.
Compute Hotelling T^2 using equation (6) as follows:

$$T^2{}_{(Quartile)i} = \big(X_i - \bar{X}_{(Quartile)}\big)\Sigma^{-1}\big(X_i - \bar{X}_{(Quartile)}\big)' \quad i = 1,2,3 \qquad (14)$$

Step 4 : Define the outliers by using UCL based on equation (9),

$$UCL \sim \chi^2_{(1-\alpha),p}$$

Step 5: Remove the observations which are considered as outliers (data that exceed UCL)
Step 6: Estimate new $\bar{X}$ and Σ using sample without outliers (obtained in Step 5 above).

Phase II

Step 1: Compute $T^2{}_{(g)} = (X_i - \bar{X}_{(Quartile)})\,\Sigma^{-1}\big(X_i - \bar{X}_{(Quartile)}\big)'$ (15)

Using sample without outliers (as obtained in Step 5 in Phase one).
where $T^2_{(g)}$ is the Hotelling T^2 statistic of the sample without outliers
Step 2: Compute UCL using equation (10)
$UCL \sim \left[\frac{P(n+1)(n-1)}{n(n-P)}\right], F_{1-\alpha}(P, n-P).$

c. Study Data

A dataset from the United States Geological Survey water use data (2014). 500 consumers and 9 quality characteristics was considered in this study which is expected to contribute to the total amount of water consumed. The listed characteristics are; total ground water withdrawals (fresh), total groundwater withdrawals (saline), total groundwater withdrawals (fresh+saline), total surface-water withdrawals (fresh), total surface-water withdrawals (fresh+saline), total withdrawals (fresh), total withdrawals (saline) and total withdrawals (fresh+saline). The choice of the real data set is being guided by the large number of outliers present in the data while the second data was obtained from a generated normal distribution contaminated with 20% outliers in order to demonstrate the performance of our proposed partition values using different mean shifts, variable number and sample sizes.

d. Performance Evaluation

The performance of the alternative technique was assessed by the probability of detection. The best Hotelling T^2 approach is expected to produce the highest probability of detection (Hanif *et al*., 2013).

e. Probability of detection

The outliers detected by Hotelling T^2 technique are denoted by, θ_i and the number of outliers will be obtained as $\theta_1, \theta_2, \dots, \theta_n$(Hanif *et al.*, 2013).
The mean number of outliers for *n* times is;

$$\bar{\theta} = \frac{\sum_{i=1}^{n} \theta_i}{1500} \qquad (16)$$

Thus, the probability of detection is obtained by dividing $\bar{\theta}$ with the sample size n (Hanif *et al*., 2013).

II. Data Analysis

An initial empirical statistic was performed on the real data set to validate the proposed approach to the Hotelling T^2 with nine (9) variables. The result of the descriptive statistic in Table 1 shows that the dataset is positively skewed and leptokurtic which implies that the distribution of the data is not normally distributed across the nine variables.

Table 1: Empirical Statistics on United States Geological Survey Water Use Data (2014)

	Statistics			
	Mean	Std Dev	Skewness	Kurtosis
Ground water-fresh	96.2684	145.88154	3.604	19.440
Ground water-saline	2.9741	9.5528	10.080	126.099
Ground water-fresh +saline	99.2426	149.2874	3.581	19.110
Surface-water of fresh	98.0740	253.8416	4.973	33.453
Surface-water of saline	-76.7993	311.73654	2.628	21.286
Surface-water of fresh + saline	21.2747	431.55416	2.321	11.250
Total Fresh	194.3425	336.54360	4.380	24.154
Total Saline	-73.8251	311.23823	2.703	21.627
Fresh +Saline	120.5174	474.10514	3.065	13.164

a. Test for Normality

Results in Table 2, shows that the two normality tests for the nine (9) variables of real data which validates that all the variables under study are not normally distributed because the probability values (p-value) are less that 5%.

Table 2 : The Normality Test of the Nine (9) Variables of United States Geological Survey water use data (2014).

Mgal/d	Kolmogorov-Smirnov			Shapiro-Wilk		
	Statistic	Df	Sig.	Statistic	Df	Sig.
Ground water fresh	0.255	500	.000	0.643	500	.000
Ground Water Saline	0.378	500	.000	0.275	500	.000
Fresh+Saline	0.253	500	.000	0.644	500	.000
Surface Water Fresh	0.350	500	.000	0.411	500	.000
Surface Water Saline	0.354	500	.000	0.567	500	.000
Surface-water (fresh + saline)	0.229	500	.000	0.749	500	.000
Total Fresh	0.282	500	.000	0.525	500	.000
Total Saline	0.339	500	.000	0.569	500	.000
Fresh +Saline	0.232	500	.000	0.689	500	.000

b. An Application of the Hotelling T^2 based on Quartile Mean on United States Geological Survey Water Data (2014)

To illustrate the usability of the proposed methods in a real life situation, we applied the existing approaches to Hotelling T^2 and the proposed approach to Hotelling T^2 based on quartile mean to United States Geological Survey Water Data (in Mgal/d). The results as shown in Table 3 below gives the probability of detection of water usage using four (4) approaches to Hotelling T^2 technique which from the results, the probability of detection value of the Hotelling T^2 (Quartile) indicates higher detection capability by 99% detection followed by Hotelling T^2 (Trimmed) with detection percentage of 25%. The traditional Hotelling T^2 and Hotelling T^2 (decile) techniques exhibited a poor performance having 8% and 2% respectively.

Table 3: shows the probability of detection using United States Geological Survey water use data (2014)

	Statistic	Percentage
P	9	
N	500	
Hotelling T^2	0.082	8%
Hotelling T^2 (Trim)	0.252	25%
Hotelling T^2 (Decile)	0.022	2%
Hotelling T^2 (quartile)	0.992	99%

Fig. 1 shows that the efficiency of traditional Hotelling T^2 technique is affected when outliers are present in the data set which has resulted to a low probability of detection of 0.082. The Hotelling T^2 (decile) equally yielded a low detection probability of 0.022 as compared to Hotelling T^2 (trim) which has a detection probability of 0.252. The proposed alternative; Hotelling T^2 (quartile) outperformed all other approaches with the highest probability of detection of 0.992.

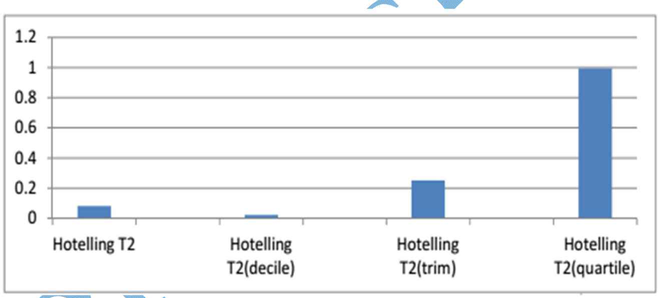

Figure 1: A plot of the probability of detection of the Hotelling T^2 techniques based on the approaches considered.

IV. Conclusion

The special feature of the Hotelling T^2 technique proposed in this study is its good outlier detection capability in the presence of numerous outliers in comparison to other approaches to Hotelling T^2 techniques. In agreement with the study carried out by; Sullivan and Woodall (1996), Shabbak and Midi (2012), (Hanif *et al*., 2013), and (Ali *et al*., 2013), the estimators of traditional Hotelling's T^2 technique are affected by an unstable process i.e multivariate outliers which has resulted to reducing the probability of detection revealing that there is a need for alternative methods to traditional Hotelling T^2 technique for more precise and accurate results especially when normality assumptions are violated. Finally, it can be concluded that the Hotelling T^2 (quartile) technique outperformed all other Hotelling T^2 approaches considered in this study and should be employed in multivariate outlier detection when outliers are present or when normality conditions are not met.

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