# A Study on Weibull-Burr XII Distribution and its Properties

**A. Yahaya[1]; M. Sa'ad[2]**

Department of Statistics,
Ahmadu Bello University,
Zaria, Nigeria.
e-mail: mohammedsaadtoro@gmail.com[2]

***Abstract*— In recent times, lots of efforts have been made to define new probability distributions that cover different aspects of human endeavours with a view to providing alternatives in modelling real data. A five-parameter distribution, called the Weibull-Burr XII (Wei-Burr XII) distribution is studied and investigated. Some of its statistical properties are obtained, these include moments, moment generating function, characteristics function, quantile function and reliability (survival) functions. The distribution's parameters are estimated by the method of maximum likelihood.**

**Keywords:** *Weibull Distribution, moments, characteristics function, reliability function, maximum likelihood estimation.*

.

## I. Introduction

Probability distributions are recently receiving many attentions with regards to introducing new generators for univariate continuous type of probability distributions by introducing additional parameter(s) to the base line distribution. This introduction is seemed necessary to reflect current realities that are not captured by the conventional probability distributions since it has been proven to be useful in exploring tail properties of the distribution under study by [10].

This idea of adding one or more parameter(s) to the baseline distribution has been in practice for a quite long time. Several distributions have been proposed in the literature to model lifetime data. Some of these distributions includes: a two-parameter exponential-geometric distribution introduced by [2] which has a decreasing failure rate. Following the same idea of the exponential geometric distribution, the exponential-Poisson distribution was introduced by Kus [8] with also a decreasing failure rate and discussed some of its properties. [9]presented a simpler technique for adding a parameter to a family of distributions with application to the exponential and Weibullfamilies. [1]Suggested the extended exponential-geometric (EEG) distribution which generalizes the exponential geometric distribution and discussed some of its statistical properties along with its hazard rate and survival functions.

Some of the well-known class of generators include the following: Kumaraswamy-G (Kw-G) proposed by [7], McDonald-G (Mc- G) introduced by [3], gamma-G type 1 presented by [11], exponentiated generalized (exp-G) which was derived by [6], others are exponentiated T-X proposed by [4]. Most recently, a New Weibull-G Family of Distributions by [10]. The Weibull–G family of probability distributions by [5]. This research is motivated by the work done by [5] - The Weibull–G family of probability distributions who introduced a generator based on the Weibull random variable called it a Weibull-G family.

In this research, we propose an extension of the pdf called the Weibull-Burr XII distribution based on the Weibull-G class of distributions defined by [5].i.e. we propose a new distribution with five parameters, referred to as the Weibull-Burr XII (Wei-BXII) distribution, which contains as special sub-models the Weibull and Burr XII distributions.

A random variable X follows the three-parameter Burr XII distribution with parameters $c, k, s > 0$ if it is respectively defined by the cumulative distribution and density function as below.

$$G(x; c, k, s) = 1 - (1 + (x/s)^c)^{-k} \qquad x \geq 0,\ c, k, s > 0 \qquad (1)$$

$$g(x; c, k, s) = cks^{-1}x^{c-1}(1 + (x/s)^c)^{-k-1} \qquad x \geq 0,\ c, k, s > 0 \qquad (2)$$

Note that k and c are shape parameters and s is a scale parameter.

Similarly, a random variable X follows the two-parameter Weibull distribution with parameters $\alpha, \beta > 0$ if it is respectively defined by the cumulative distribution and density function as below.

$$G(x; \alpha, \beta) = 1 - e^{-\alpha x^{\beta}} \qquad \alpha, \beta > 0, x > 0 \qquad (3)$$

$$g(x;\alpha,\beta)=\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}},\quad \alpha,\beta>0,x>0 \tag{4}$$

Where, $\alpha>0$ is the shape parameter and $\beta>0$ is the scale parameter of the distribution.

The cumulative distribution function of the Weibull-G Family of Distribution defined by Bourguignon *et al.*, (2014) is given by;

$$G(x;\alpha,\beta,\boldsymbol{\xi})=1-e^{-\alpha\left[\frac{G(x;\xi)}{1-G(x;\xi)}\right]^{\beta}},\qquad x\subseteq R,\alpha,\beta>0 \tag{5}$$

Where the random variable x in equation (3) is replaced with $\frac{G(x;\xi)}{1-G(x;\xi)}$ and $G(x;\boldsymbol{\xi})$is the CDF of the baseline distribution which depends on a parameter vector $\boldsymbol{\xi}$. The family pdf reduces to

$$g(x;\alpha,\beta,\boldsymbol{\xi})=\alpha\beta g(x;\boldsymbol{\xi})\frac{G(x;\xi)^{\beta-1}}{1-G(x;\xi)^{\beta+1}}exp\left\{-\alpha\left[\frac{G(x;\xi)}{1-G(x;\xi)}\right]^{\beta}\right\}x\subseteq R,\alpha,\beta>0 \tag{6}$$

The paper is arranged as follows: In section 2, we defined the pdf, cdfand the reliability functions of the proposedWeibull-BurrXII distribution. Section 3 provides some mathematical properties of the proposed distribution. In section 4, estimates of the parameters of the new distribution were presented. Finally in section 5, a brief concluding remark was obtainable.

## II. Research Methodology

### A. *PDF, CDF, and Reliability functions of Weibull-Burr XII distribution*

A random variable X follows the proposed five-parameter Weibull-Bur XII distribution with parameters $\alpha,\beta,c,k,s>0$ if it is respectively defined by the cumulative distribution function and probability density function as below.

$$G(x;\alpha,\beta,c,k,s)=1-exp\left(-\alpha\left((1+(x/s)^{c})^{k}-1\right)^{\beta}\right),\quad \alpha,\beta,c,k,s>0,\quad x\geq 0 \tag{7}$$

$$g(x;\alpha,\beta,c,k,s)=\alpha\beta\,cks^{-c}x^{c-1}\times(1+(x/s)^{c})^{k-1}\times\left((1+(x/s)^{c})^{k}-1\right)^{\beta-1}exp\left\{-\alpha\left[\left((1+(x/s)^{c})^{k}-1\right)\right]^{\beta}\right\},\ \alpha,\beta,c,k,s>0,x\geq 0 \tag{8}$$

Equations (7) and (8) are obtained by substituting equations (1) and (2) into equation (6).

Reliability functions include Survival function and hazard rate function which are respectively obtainable using the formulae as below.

$$S(x)=\Pr(x>0)=1-G(x)$$

$$h(x)=\frac{g(x)}{1-G(x)}$$

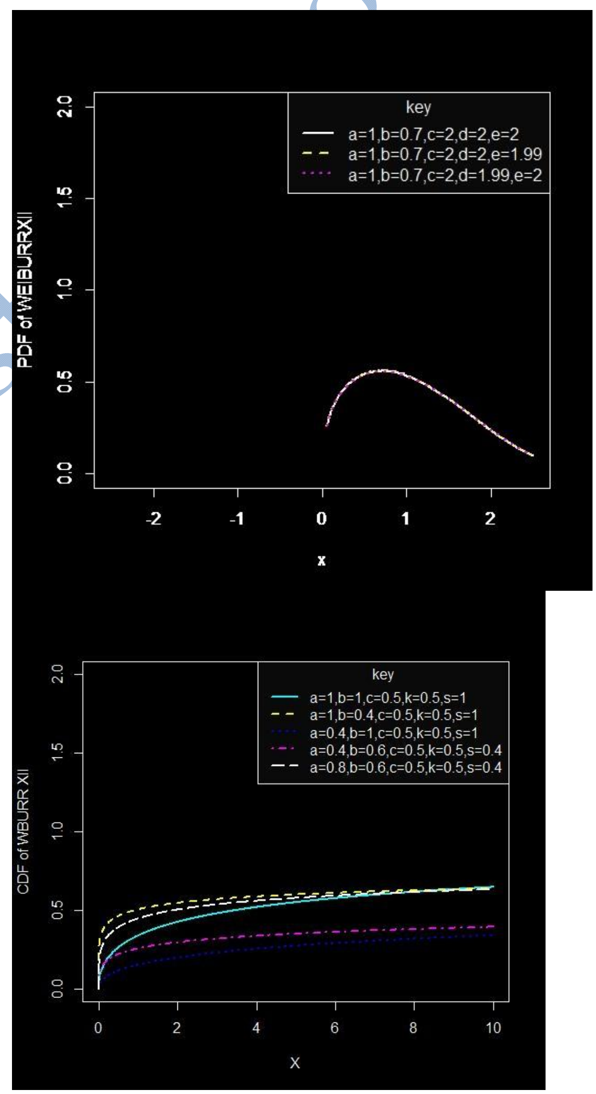
where$g(x)$ and $G(x)$ is the probability density function and cumulative distribution function of a distribution respectively.

The Survival and hazard rate functions of the Weibull-Burr XII distribution is respectively defined by

$$S(x)=exp\left(-\alpha\left((1+(x/s)^{c})^{k}-1\right)^{\beta}\right),\qquad x\geq 0 \tag{9}$$

$$h(x;\alpha,\beta,c,k,s)=\alpha\beta cks^{-c}x^{c-1}(1+(x/s)^{c})^{k-1}\left((1+(x/s)^{c})^{k}-1\right)^{\beta-1},\ x\geq 0 \tag{10}$$

**Plots of the CDF and PDF of Weibll-Burr XII distribution.**




### B. *Statistical Properties of Weibull-Burr XII distribution*

3.1 *Quantile Function*

Quantile function is defined as
$Q(u) = G^{-1}(u)$ (11)

$$Q(u) = \sqrt[c]{\left[s^c\left\{1+\left(\frac{1}{\alpha}In\left(\frac{1}{1-u}\right)^{1/\beta}\right)\right\}^{1/k} - 1\right]}$$ (12)

**Proof:** Let

$$G(x;\alpha,\beta,c,k,s) = 1 - exp\left(-\alpha\left((1+(x/s)^c)^k - 1\right)^\beta\right) = u$$

$$\Rightarrow \quad exp\left(-\alpha\left((1+(x/s)^c)^k - 1\right)^\beta\right) = 1-u$$

$$\left((1+(x/s)^c)^k - 1\right)^\beta = \frac{1}{\alpha}In\left(\frac{1}{1-u}\right)$$

$$\frac{x^c}{s^c} = \left\{1+\left(\frac{1}{\alpha}In\left(\frac{1}{1-u}\right)^{1/\beta}\right)\right\}^{1/k} - 1$$

$$x^c = s^c\left\{1+\left(\frac{1}{\alpha}In\left(\frac{1}{1-u}\right)^{1/\beta}\right)\right\}^{1/k} - 1$$

Therefore,

$$Q(u) = x = \sqrt[c]{\left[s^c\left\{1+\left(\frac{1}{\alpha}In\left(\frac{1}{1-u}\right)^{1/\beta}\right)\right\}^{1/k} - 1\right]}$$

**3.2 Series Expansion of the pdf of Weibull-Burr XII distribution**

Remember the power series expansion of

$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}$ and $e^{-cx} = \sum_{k=0}^{\infty}\frac{(-1)^k c^k}{k!}x^k$ (13)

Expanding the exponential term in equation (8) by using power series expansion, we have

$$exp\left\{-\alpha\left[\left((1+(x/s)^c)^k - 1\right)\right]^\beta\right\} = \sum_{k=0}^{\infty}\frac{(-1)^k\alpha^k}{k!}\left((1+(x/s)^c)^k - 1\right)^{k\beta} \quad (14)$$

Substituting equation (14) into equation (8) above, we get

$$g(x;\alpha,\beta,c,k,s) = \alpha\beta\, cks^{-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{j+k}\alpha^k\Gamma(k\beta+\beta)}{j!\,k!\,\Gamma(k\beta+\beta-j)}\left(\frac{x}{s}\right)^{c-1}\left(1+(x/s)^c\right)^{k(k\beta+\beta-j)-1}$$

so that the reduced version becomes

$$g(x;\alpha,\beta,c,k,s) = W_{j,k}x^{c-1}\left(1+(x/s)^c\right)^{k(k\beta+\beta-j)-1} \quad (15)$$

Where

$$W_{j,k} = \alpha\beta\, cks^{-c}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{j+k}\alpha^k\Gamma(k\beta+\beta)}{j!k!\Gamma(k\beta+\beta-j)} \quad (16)$$

**3.3Moments**

The nth moment of a random variable X following a Weibull-Burr XII distribution with parameters $\alpha,\beta,c,k,s > 0$ is given by

$$E(x^n) = W_{j,k}\frac{s^{c+n}(-1)^{\frac{n}{c}}}{c}\beta\left(k(k\beta+\beta-j), \frac{n}{c}+1\right)$$

where $W_{j,k}$ as in equation (17)

**Proof:** The nth moment of a random variable X having a pdf $g(x)$ is given by

$$E(x^n) = \int_0^\infty x^n g(x)dx \quad (17)$$

By substituting equation (16) into (18), we get

$$E(x^n) = W_{j,k}\int_0^\infty x^{n+c-1}\left(1+(x/s)^c\right)^{k(k\beta+\beta-j)-1}dx \quad (18)$$

Now, let $y = 1+\frac{x^c}{s^c} \Rightarrow \frac{dy}{dx} = \frac{cx^c}{s^c}$ and $dx = \frac{s^c dy}{cx^{-c}}$

Also, let $y = 1+\frac{x^c}{s^c} \Rightarrow x = (-1)^{\frac{1}{c}}s(1-y)^{\frac{1}{c}}$

Substituting for y and dx into equation (19) we obtain

$$E(x^n) = W_{j,k}\int_0^\infty\frac{s^c}{c}y^{k(k\beta+\beta-j)-1}\frac{s^c}{cx^{c-1}}dy$$

$$E(x^n) = W_{j,k}\int_0^\infty\frac{s^c}{c}x^{n+c-1-(c-1)}y^{k(k\beta+\beta-j)-1}dy$$

$$E(x^n) = W_{j,k}\frac{s^c}{c}\int_0^\infty x^n\, y^{k(k\beta+\beta-j)-1}dy \quad (19)$$

Recall that, $x = (-1)^{\frac{1}{c}}s(1-y)^{\frac{1}{c}}$

Substituting for x into equation (20) gives;

$$E(x^n) = W_{j,k}\frac{s^c}{c}\int_0^\infty\left[(-1)^{\frac{1}{c}}s(1-y)^{\frac{1}{c}}\right]^n y^{k(k\beta+\beta-j)-1}dy$$

$$= W_{j,k}\frac{s^{c+n}(-1)^{\frac{n}{c}}}{c}y^{k(k\beta+\beta-j)-1}(1-y)^{\frac{n}{c}}$$

$$= W_{j,k}\frac{s^{c+n}(-1)^{\frac{n}{c}}}{c}y^{k(k\beta+\beta-j)-1}(1-y)^{\frac{n}{c}+1-1}dy$$

Recall that the beta function is expressed as

$$\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$$

Hence,

$$E(x^n) = W_{j,k}\frac{s^{c+n}(-1)^{\frac{n}{c}}}{c}\beta\left(k(k\beta+\beta-j), \frac{n}{c}+1\right)$$

This completes the proof.

**3.4. Moment generating function**

The moment generating function (mgf) of a random variable X that follows Weibull-Burr XII distribution is given by

$M_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E(x^n)$ (20)

**Proof:**
The moment generating function (mgf) of a random variable X following a pdf $g(x)$ is given by;

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} g(x)dx$$

Using Maclaurin's power series expansion, $e^{tx} = \sum_{n=0}^{\infty} \frac{t^n}{n!} x^n$

It implies that

$$M_X(t) = \int_0^{\infty} \sum_{n=0}^{\infty} \frac{t^n}{n!} x^n g(x;\alpha,\beta,c,k,s)dx$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_0^{\infty} x^n g(x;\alpha,\beta,c,k,s)dx$$

$$M_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_0^{\infty} x^n g(x;\alpha,\beta,c,k,s)dx$$

Therefore, $M_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E(x^n)$

**3.4 Characteristic Function**

The characteristic function of a random variable X that follows Weibull-Burr XII distribution is defined by;

$$\phi_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{(2n)!} E(x^{2n}) + i \sum_{n=0}^{\infty} \frac{t^n}{(2n+1)!} E(x^{2n+1})$$

**Proof:**
The characteristic function of a random variable x having a pdf $g(x)$ is defined by;

$$\phi_X(t) = E(e^{itx}) g(x)dx \quad (21)$$

Recall that, $e^{itx} = cos(tx) - isin(tx)$

Using series expansion, it is known that

$\cos(tx) = \sum_{n=0}^{\infty} \frac{t^n}{(2n)!} x^{2n}$ and $\sin(tx) = \sum_{n=0}^{\infty} \frac{t^n}{(2n+1)!} x^{2n+1}$

Therefore, substituting equation (16) into equation (22), we get

$$\phi_X(t) = \int_0^{\infty} \left( \sum_{n=0}^{\infty} \frac{t^n}{(2n)!} x^{2n} + i \sum_{n=0}^{\infty} \frac{t^n}{(2n+1)!} x^{2n+1} \right) g(x;\alpha,\beta,c,k,s)\, dx$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{(2n)!} \int_0^{\infty} x^{2n} g(x;\alpha,\beta,c,k,s)dx + i \sum_{n=0}^{\infty} \frac{t^n}{(2n+1)!} \int_0^{\infty} x^{2n+1} g(x;\alpha,\beta,c,k,s)dx$$

Hence,

$$\phi_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{(2n)!} E(x^{2n}) + i \sum_{n=0}^{\infty} \frac{t^n}{(2n+1)!} E(x^{2n+1})$$

## III. PARAMETER ESTIMATION FOR WEIBULL-BURR XII DISTRIBUTION

At this point, we study the estimation of the unknown parameters of the Weibull-Burr XII distribution by the method of maximum likelihood. Let $x_1, x_2, \dots, x_n$ be a sample of size n following a Weibull-Burr XII distribution given by (16). The likelihood and log-likelihood functions for the parameters $\Theta = (\alpha,\beta,c,k,s)^T$ can be expressed as

$$l(X/\alpha,\beta,c,k,s) = (\alpha\beta cks^{-c})^n \sum_{i=1}^{n} \left(1+\left(\frac{x_i}{s}\right)^c\right)^{k-1} \sum_{i=1}^{n} x_i^{\,c-1} \sum_{i=1}^{n} \left[\left(1+\left(\frac{x_i}{s}\right)^c\right)^k - 1\right]^{\beta-1} \times exp\left\{-\alpha \sum_{i=1}^{n} \left[\left(1+\left(\frac{x_i}{s}\right)^c\right)^k - 1\right]^{\beta}\right\} \quad (22)$$

and

$$l(\Theta) = n\log\alpha + \mathrm{n}\log\beta + n\log c + \log k - cn\log s + (k-1)\sum_{i=1}^{n} \log\left(1+\left(\frac{x_i}{s}\right)^c\right) + (c-1)\sum_{i=1}^{n}\log(x_i) + (\beta-1)\sum_{i=1}^{n} \log\left(\left(1+\left(\frac{x_i}{s}\right)^c\right)^k - 1\right) - \alpha\sum_{i=1}^{n}\left(\left(1+\left(\frac{x_i}{s}\right)^c\right)^k - 1\right)^{\beta} \quad (23)$$

respectively.

The components of the vector $U(\Theta)$ are given by taking the partial derivatives of equation (24) w.r.t. the parameter involved and equating to zero, we obtain

$$U_\alpha = \frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \left(\left(1+\left(\frac{x_i}{s}\right)^c\right)^k - 1\right)^{\beta} = 0$$

$$U_\beta = \frac{\partial l}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \log\left(\left(1+\left(\frac{x_i}{s}\right)^c\right)^k - 1\right) - \sum_{i=1}^{n} \left(\left(1+\left(\frac{x_i}{s}\right)^c\right)^k - 1\right)^{\beta} \log\left(\left(1+\left(\frac{x_i}{s}\right)^c\right)^k - 1\right) = 0$$

$$U_c = \frac{\partial l}{\partial c} = \frac{n}{c} - n\log(s) + \sum_{i=1}^{n} \log(x_i) + (k-1)\sum_{i=0}^{n} \left\{ \frac{\left(\frac{x_i}{s}\right)^c ln\left(\frac{x_i}{s}\right)}{1+\left(\frac{x_i}{s}\right)^c} \right\}$$

$$+k(\beta-1)\sum_{i=0}^{n}\left\{\frac{\left(1+\left(\frac{x_i}{s}\right)^c\right)^{k-1}\left(\frac{x_i}{s}\right)^c ln\left(\frac{x_i}{s}\right)}{\left(1+\left(\frac{x_i}{s}\right)^c\right)^k-1}\right\}$$

$$-k\alpha\beta\sum_{i=0}^{n}\left\{\left[\left(1+\left(\frac{x_i}{s}\right)^c\right)^k-1\right]^{\beta-1}\left(1+\left(\frac{x_i}{s}\right)^c\right)^k\left(\frac{x_i}{s}\right)^c ln\left(\frac{x_i}{s}\right)\right\}=0$$

$$U_k=\frac{\partial l}{\partial k}=\frac{n}{k}+\sum_{i=0}^{n}\log\left(1+\left(\frac{x_i}{s}\right)^c\right)+k(\beta-1)\sum_{i=0}^{n}\left\{\frac{\left(1+\left(\frac{x_i}{s}\right)^c\right)^{k-1}}{\left(1+\left(\frac{x_i}{s}\right)^c\right)^k-1}\right\}$$

$$-k\alpha\beta\sum_{i=0}^{n}\left\{\left[\left(1+\left(\frac{x_i}{s}\right)^c\right)^k-1\right]^{\beta-1}\left(1+\left(\frac{x_i}{s}\right)^c\right)^{k-1}\right\}=0$$

$$U_s=\frac{\partial l}{\partial s}=\frac{cn}{k}-c(k-1)\sum_{i=0}^{n}\left\{\frac{x_i{}^c}{\left(1+\left(\frac{x_i}{s}\right)^c\right)s^{c+1}}\right\}-ck(\beta-1)\sum_{i=0}^{n}\left\{\frac{x_i{}^c\left(1+\left(\frac{x_i}{s}\right)^c\right)^{k-1}}{\left(\left(1+\left(\frac{x_i}{s}\right)^c\right)^k-1\right)s^{c+1}}\right\}$$

$$+\frac{ck\alpha\beta}{s^{c+1}}\sum_{i=0}^{n}\left\{x^c\left[\left(1+\left(\frac{x_i}{s}\right)^c\right)^k-1\right]^{\beta-1}\left(1+\left(\frac{x_i}{s}\right)^c\right)^{k-1}\right\}=0$$

Hence, the MLEs are obtainable by solving these equations simultaneously which is very complicated and could lead to errors, therefore use of statistical soft wares to solve the equations numerically is recommended.

## V. Conclusion

We define a new five-parameter distribution called Weibull-Burr XII distribution. Some statistical properties are derived and the pdf and cdf plots are presented. We give short form expressions for the moments of the distribution. Estimation by method of maximum likelihood is studied.

.

## References

[1] Adamidis, K., Dimitrakopoulou, T., Loukas, S. (2005); On a generalization of the exponential-geometric distribution. *Statist. and Probab. Lett.*, 73, 259-269.

[2] Adamidis, K., Loukas, S. (1998); A lifetime distribution with decreasing failure rate. *Statist. and Probab. Lett.*, 39, 35-42.

[3] Alexander, C. (2012); Generalized beta-generated distributions. *Computational Statistics and Data Analysis,* 56, 1880–1897.

[4] Alzaghal, A., Famoye, F. and Lee, C. (2013); Exponentiated T-X family of distributions with some applications. *International Journal of Probability and Statistics ,2*, 31–49.

[5] Bourguignon, M., Silva, R. B. and Cordeiro, G. M. (2014); The Weibull–G family of probability distributions. *Journal of Data Science,* 12, 53–68.

[6] Cordeiro, G.M., Ortega, E.M.M. and Cunha, D.C.C. (2013); The exponentiated generalized class of distributions. *Journal of Data Science,* 11, 1–27.

[7]Cordeiro, G.M. and de Castro, M. (2011); A new family of generalized distributions. *Journal of Statistical Computation and Simulation,* 81, 883–893.

[8] Kus, C., (2007); A new lifetime distribution. *Computat.Statist. Data Analysis*, 51, 4497-4509.

[9] Marshall, A.W., Olkin, I., (1997); A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, 84, 641-652.

[10] Tahir, M. H., Zubair, M., Mansoor, M., Gauss, M. C., Morald, A. and Hamedani, G. G. (2016); A New Weibull-G Family of Distributions. *Hacettepe Journal of Mathematics and Statistics*, Vol.45(2)629-647.

[11] Zografos K. and Balakrishnan, N. (2009); On families of beta and generalized gamma-generat distributions and associated inference, *Statistical Methodology* 6, 344–362.

.

.