A Note On The Transmuted Weibull-Rayleigh Distribution

Abubakar Yahaya; Terna Godfrey Ieren

Department of Statistics,
Ahmadu Bello University,
Zaria, Nigeria.
e-mail: abubakaryahaya@abu.edu.ng; ternagodfrey@gmail.com

***Abstract*—In this paper we study a three-parameter probability distribution entitled "Transmuted Weibull-Rayleigh Distribution (*TWRD*)" which is a generalization of the Weibull-Rayleigh distribution introduced by [6]. The definition is made possible by using the approach of Quadratic Rank Transmutation Map (QRTM) proposed by [9]. Some properties of the proposed distribution are discussed; we also provided explicit expressions for its quantile function, Renyi entropy, survival function, and hazard function. It was observed that, the probability density takes many forms such as symmetric, asymmetric, unimodal or bimodal depending on the values of the parameters. Some of the results presented in this work revealed that the survival function is decreasing, while the hazard function is increasing and their behavior depends on the values taken by the parameters.**

***Keywords**-Quadratic rank transmutation map, Transmuted Weibull-Rayleigh distribution, Quantile, survival function, hazard function, Renyi entropy.*

I. INTRODUCTION

The Rayleigh distribution has a wide range of applications including life testing experiments, reliability analysis, applied statistics and clinical studies. This distribution is a special case of the two parameter Weibull distribution with the shape parameter equal to 2. It was derived by [8] from the amplitude of sound resulting from many important sources. Recently, there has been several generalizations of the Rayleigh distribution introduced by some researchers such as, the generalized Rayleigh distribution by [4], Bivariate generalized Rayleigh distribution by [1], Transmuted Rayleigh distribution by [5], the Weibull-Rayleigh distribution by [6] and the Transmuted inverse Rayleigh distribution by [2]. These distributions have been found to be more flexibly than the Rayleigh distribution when applied to real life datasets.

According to [6], if X denotes a random variable, then the probability density function (*pdf*) and the cumulative distribution function (*cdf*) of a Weibull-Rayleigh distribution are respectively given by

$$g(x)=\alpha\beta\theta x e^{\frac{\theta}{2}x^2}\left(e^{\frac{\theta}{2}x^2}-1\right)^{\beta-1} e^{-\alpha\left(e^{\frac{\theta}{2}x^2}-1\right)^{\beta}} \qquad (1)$$

as the pdf, and

$$G(x)=1-e^{-\alpha\left(e^{\frac{\theta}{2}x^2}-1\right)^{\beta}} \qquad (2)$$

as the cdf. where; α and θ are scale parameters and β is shape parameter.
The aim of this paper is to obtain the *TWRD* by using the *QRTM* proposed by [9].

II. RESEARCH METHODOLOGY

The *TWRD*
A random variable X is said to have a transmuted distribution function if its *pdf* and *cdf* are respectively given by;

$$f(x)=g(x)\left[1+\lambda-2\lambda G(x)\right] \qquad (3)$$

and

$$F(x)=(1-\lambda)G(x)-\lambda\left[G(x)\right]^2 \qquad (4)$$

where; $x>0$, and $-1\leq\lambda\leq 1$ is the transmuted parameter
G(x) is the *cdf* of any continuous distribution while *f(x)* and *g(x)* are the associated *pdf* of *F(x)* and *G(x),* respectively.
Hence, using equations (3) and (4) above, we defined the *cdf* and *pdf* a *TWRD* with parameters α, β, θ and λ as;

$$F(x)=1-(1-\lambda)e^{-\alpha\left(e^{\frac{\theta}{2}x^{2}}-1\right)^{\beta}}-\lambda e^{-2\alpha\left(e^{\frac{\theta}{2}x^{2}}-1\right)^{\beta}} \quad (5)$$

and

$$f(x)=\alpha\beta\theta x e^{\frac{\theta}{2}x^{2}}\left(e^{\frac{\theta}{2}x^{2}}-1\right)^{\beta-1}e^{-\alpha\left(e^{\frac{\theta}{2}x^{2}}-1\right)^{\beta}}\left[1-\lambda+2\lambda e^{-\alpha\left(e^{\frac{\theta}{2}x^{2}}-1\right)^{\beta}}\right] \quad (6)$$

respectively.
For $x>0$, $\alpha, \beta, \theta>0$ and $-1 \le \lambda \le 1$, where α and θ are the scale parameters, β is the shape parameter and λ is the transmuted parameter.
At different values of the parameters α, β, θ and λ, we provide some possible shapes for the *pdf* and *cdf* of the *TWRD* as shown in figure 1 and 2 below:

I. ANALYSIS

In this section, we study some statistical properties of TWRD as follows:

A The Quantile function for the TWRD

The Quantile function plays a very important role in the calculation of moments of random variables such as skewness and kurtosis. It is also used to obtain the median and for simulation of random numbers. Mathematically, it is obtained as the inverse of the *cdf*.
Hence, quantile function, say $X=Q(u)$, of the *TWRD* can be obtained by taking the inverse of Equation (5) as;

$$Q(u)=X_{q}=\sqrt{\frac{2}{\theta}\ln\left[\left(\frac{1}{\alpha}\ln(1-u)\right)^{\frac{1}{\beta}}\right]} \quad (7)$$

Hence, the median of X from the *TWRD* is simply $X_{0.5}=Q(0.5)$ is derived by setting u=0.5 in equation (7). Furthermore, it is possible to generate *TWRD* variates by setting $X=Q(u)$, where u is a uniform variate on the unit interval *(0,1)*. The lower and the upper quartile can also be derived from (7) by setting u=0.25 and u=0.75 respectively.

B The Skewness and kurtosis

This section of our paper provides the classical measures of skewness and kurtosis because there are many heavy tailed distributions for which these measures are infinite and therefore, it becomes uninformative precisely when they are highly needed. The Bowley's Skewness [3] based on quartile is defined as;

$$B=\frac{Q\left(\frac{3}{4}\right)+Q\left(\frac{1}{4}\right)-2Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right)-Q\left(\frac{1}{4}\right)} \quad (8)$$

And the Moor's Kurtosis [7] based on Octiles is given by;

$$M=\frac{Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)+Q\left(\frac{3}{8}\right)-Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)} \quad (9)$$

where $Q(.)$ represents the quantile function.

C Reliability analysis of the TWRD

The survival function, also known as the reliability function in engineering, is the characteristic of an explanatory variable that maps a set of events, usually associated with mortality or failure of some system onto time. It is the probability that the system will survive beyond a specified time. Mathematically, the survival function is given by;

$$S(X)=P(X>x)=1-F(x) \quad (10)$$

Therefore, the survival function for the *TWRD* can be simplified to give;

$$S(X)=(1-\lambda)e^{-\alpha\left(e^{\frac{\theta}{2}x^{2}}-1\right)^{\beta}}+\lambda e^{-2\alpha\left(e^{\frac{\theta}{2}x^{2}}-1\right)^{\beta}} \quad (11)$$

For $x>0$, where $\alpha, \beta, \theta>0$ and $|\lambda| \le 1$.
For brevity, a plot for the survival function of the *TWRD* at different values the parameters is as shown in Figure 4.

The hazard function is defined as the probability per unit time that a case which has survived to the beginning of the respective interval will fail in that interval. Specifically, it is computed as the number of failures per unit time in the respective interval, divided by the average number of surviving cases at the mid-point of the interval. Mathematically, the hazard function for a random variable X is defined as:

$$h(x)=\frac{f(x)}{S(x)}=\frac{f(x)}{1-F(x)} \quad (12)$$

Hence, the expression for the hazard rate of the *TWRD* is given by

$$h_{TWR}(x)=\frac{\alpha\beta\theta x e^{\frac{\theta}{2}x^{2}}\left(e^{\frac{\theta}{2}x^{2}}-1\right)^{\beta-1}\left[1-\lambda+2\lambda e^{-\alpha\left(e^{\frac{\theta}{2}x^{2}}-1\right)^{\beta}}\right]}{1-\lambda+\lambda e^{-\alpha\left(e^{\frac{\theta}{2}x^{2}}-1\right)^{\beta}}} \quad (13)$$

where $\alpha, \beta, \theta > 0$ and $|\lambda| \le 1$.

A possible plot for the hazard rate at various values of parameter α,β,θ and λ are shown in Figures 5 below:

D Entropy

Entropy is a function used to quantify the uncertainty, disorderliness or randomness in a system or a probability distribution. The Ren'yi entropy of a random variable X represents a variation of the uncertainty. It is defined by;

$$I_\delta(X)=\frac{1}{1-\delta}\log\int_{-\infty}^{\infty}f^{\delta}(x)\,dx \quad (14)$$

For $\delta>0$ and $\delta\neq 1$

Now, for the *TWRD* we have

$$f^{\delta}(x)=\sum_{i,j,k=0}^{\infty}(-1)^k\binom{\beta(i+\delta)+j-\delta}{k}\beta\theta x^{\delta}\, e^{-(\delta+k)\frac{\theta}{2}x^2}\left[(1-\lambda)Wi,j+2\lambda W^{*}_{i,j}\right] \quad (15)$$

Where $W_{i,j}=\dfrac{(-1)^i\,\delta^i\alpha^{i+1}\Gamma\left(\beta(i+\delta)+j-\delta\right)}{i!\,j!\,\Gamma\left(\beta(i+\delta)+\delta\right)}$ and

$$W^{*}_{i,j}=\frac{(-2)^i\,\delta^i\alpha^{i+1}\Gamma\left(\beta(i+\delta)+j-\delta\right)}{i!\,j!\,\Gamma\left(\beta(i+\delta)+\delta\right)}$$

Finally, the Renyi entropy can be written as

$$I_\delta(X)=\frac{1}{1-\delta}\log\left[\alpha\beta\theta\sum_{i,j,k=0}^{\infty}(-1)^k\binom{\beta(i+\delta)+j-\delta}{k}\right.$$

$$\left.\left[(1-\lambda)W_{i,k}+2\lambda W^{*}_{i,k}\right]\right]\frac{2^{\frac{\delta-1}{2}}\Gamma\left(\frac{\delta+1}{2}\right)}{\left[\theta(\delta+k)\right]^{\frac{\delta+1}{2}}} \quad (17)$$

E Estimation of Parameters of the *TWRD*.

Let $X_1,X_2,\ldots\ldots\ldots,X_n$ be a sample of size 'n' independently and identically distributed random variables from the *TWRD* with unknown parameters α, β, θ, and λ defined previously. The likelihood function is given by;

$$L(X_1,X_2,\ldots,X_n/\alpha,\beta,\theta,\lambda)=$$

$$(\alpha\beta\theta)^n\sum_{i=1}^{n}x_i e^{\frac{\theta}{2}\sum_{i=1}^{n}x_i^2}\sum_{i=1}^{n}\left(e^{\frac{\theta}{2}x_i^2}-1\right)^{\beta-1}e^{-\alpha\sum_{i=1}^{n}\left(e^{\frac{\theta}{2}x_i^2}-1\right)^{\beta}}\prod_{i=1}^{n}\left[1-\lambda+2\lambda e^{-\alpha\left(e^{\frac{\theta}{2}x_i^2}-1\right)^{\beta}}\right]$$

Let the log-likelihood function $l=\log L(X_1,X_2,\ldots,X_n\,|\,\alpha,\beta,\theta,\lambda)$, therefore

$$l=n\log\alpha+n\log\beta+n\log\theta+\sum_{i=1}^{n}\log(x_i)+\frac{\theta}{2}\sum_{i=1}^{n}x_i^2$$

$$+(\beta-1)\sum_{i=1}^{n}\log\left(e^{\frac{\theta}{2}x_i^2}-1\right)-\alpha\sum_{i=1}^{n}\left(e^{\frac{\theta}{2}x_i^2}-1\right)^{\beta}\sum_{i=1}^{n}\log\left[1-\lambda+2\lambda e^{-\alpha\left(e^{\frac{\theta}{2}x_i^2}-1\right)^{\beta}}\right] \quad (19)$$

Differentiating $l(19)$ partially with respect to α, β, θ and λ respectively gives;

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\left(e^{\frac{\theta}{2}x_i^2}-1\right)^{\beta}-2\lambda\sum_{i=1}^{n}\left\{\frac{\left(e^{\frac{\theta}{2}x_i^2}-1\right)^{\beta}e^{-\alpha\left(e^{\frac{\theta}{2}x_i^2}-1\right)^{\beta}}}{1-\lambda+2\lambda e^{-\alpha\left(e^{\frac{\theta}{2}x_i^2}-1\right)^{\beta}}}\right\} \quad (20)$$

$$\frac{\partial l}{\partial\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\log\left(e^{\frac{\theta}{2}x_i^2}-1\right)^{\beta}-\alpha\sum_{i=1}^{n}\left(e^{\frac{\theta}{2}x_i^2}-1\right)^{\beta}\log\left(e^{\frac{\theta}{2}x_i^2}-1\right)-2\lambda\alpha\sum_{i=1}^{n}\left\{\frac{\left(e^{\frac{\theta}{2}x_i^2}-1\right)^{\beta-1}\log\left(e^{\frac{\theta}{2}x_i^2}-1\right)e^{-\alpha\left(e^{\frac{\theta}{2}x_i^2}-1\right)^{\beta}}}{1-\lambda+2\lambda e^{-\alpha\left(e^{\frac{\theta}{2}x_i^2}-1\right)^{\beta}}}\right\}$$

(21)

$$\frac{\partial l}{\partial\theta}=\frac{n}{\theta}+\frac{1}{2}\sum_{i=1}^{n}x_i+\frac{(\beta-1)}{2}\sum_{i=1}^{n}\left\{\frac{x_i^2e^{\frac{\theta}{2}x_i^2}}{\left(e^{\frac{\theta}{2}x_i^2}-1\right)}\right\}-\frac{\alpha\beta}{2}\sum_{i=1}^{n}\left\{x_i^2e^{\frac{\theta}{2}x_i^2}\left(e^{\frac{\theta}{2}x_i^2}-1\right)^{\beta-1}\right\}$$

(22)

$$\frac{\partial l}{\partial\lambda}=\sum_{i=1}^{n}\left\{\frac{2e^{-\alpha\left(e^{\frac{\theta}{2}x_i^2}-1\right)^{\beta}}-1}{1-\lambda+2\lambda e^{-\alpha\left(e^{\frac{\theta}{2}x_i^2}-1\right)^{\beta}}}\right\} \quad (23)$$

To obtain the Maximum likelihood estimates, $\hat{\alpha},\hat{\beta},\hat{\theta},and,\hat{\lambda}$, we equate (20), (21), (22) and (23) to zero and solve. The solution of the non-linear system of the above equations gives the maximum likelihood estimates of parameters α,β,θ and λ. However, the solution cannot be possible analytically except numerically with the aid of suitable statistical software like Python, *R*, *SAS*, *e.t.c* when data sets are available due to the nature of the equations.

II. RESULT

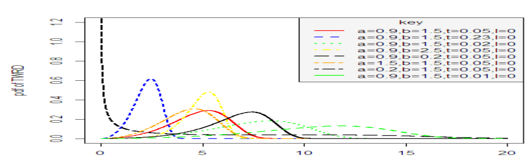

Fig. 1: The graph of the *PDF* of the *TWRD* for different parameter values $a=\alpha,b=\beta,t=\theta andl=\lambda=0.$

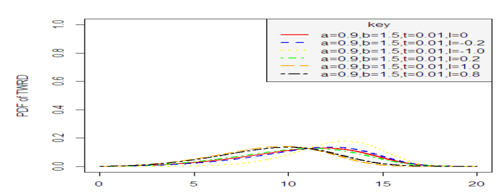

Fig. 2: The graph of the *PDF* of the *TWRD* for different parameter values of λ where $a=\alpha,b=\beta,t=\theta andl=\lambda.$

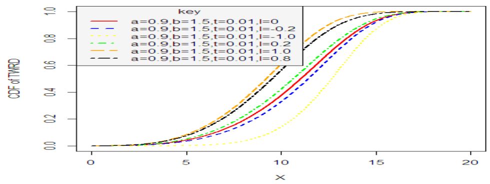

Fig. 2: The graph of the *CDF* of the *TWRD* for different parameter values

$$a = \alpha, b = \beta, t = \theta and l = \lambda.$$

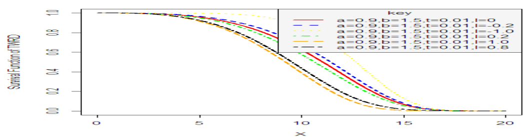

Fig. 4: The graph for the Survival function of the *TWRD*for different values of λ and some values of the parameters where

$$a = \alpha, b = \beta, t = \theta and l = \lambda.$$

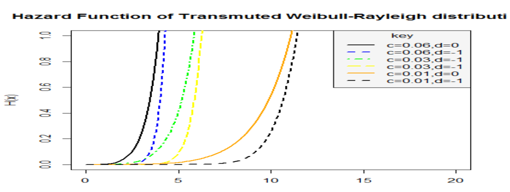

Fig. 5: Hazard function of the *TWRD* for different values of λ and some values of the parameters where $\alpha = 2.5, \beta = 2.5, c = \theta and d = \lambda.$

III. DISCUSSIONS

Figure 1 and 2 above show that the *TWRD* has various shapes such as symmetrical, left-skewed, right-skewed, and reversed-J shapes, which is an indication that the *TWRD* can be used to model datasets with various shapes. The graph of *cdf* of the *TWRD* above shows that the *cdf* increases when X increases, and approaches 1 when X becomes large, as it is well known. We can see from figure 4 above that the value of the survival function equals one (1) at initial time or early age and it decreases as X increases and remains constant as X equals zero (0). The implication of this behavior explains that the *TWRD* will be useful in modeling time or age-dependent events, where the probability of life or success decreases with time or age, that is, it gets smaller as time goes on till it reaches zero.

The graph in figure 5 above illustrate that the probability of failure for any variable X following the *TWRD* will be high at initial time or it's early age but decreases when X increases. It gets smaller as the value of X increases. The reason for this behavior is that the *TWRD* may be appropriate in modeling time or age-dependent events, where risk or hazard decreases with time or age. Many examples are found in systems of components that fail as a result of the age of those components. Figures **5** at other parameter values also revealed that this distribution can produce hazard rate shapes such as increasing, decreasing, J, reversed-J, bathtub and upside-down bathtub as an indication that it can be applied in many situations where the data set has different shapes.

IV. CONCLUSION

This paper studies some mathematical and statistical properties of a newly proposed distribution called the *TWRD*. We have derive explicit expressions for its survival function, hazard function, Ren'yi entropy and quantile function which is useful for obtaining the median, skewness and kurtosis and simulation of random numbers from the *TWRD*. Some plots of the distribution revealed that it takes various shape and could model different types of data sets. Some of the results presented in this work revealed that the survival function is decreasing, while the hazard function is increasing and their behavior depends on the values taken by the parameters. We estimated the model parameters using the method of maximum likelihood estimation. Based on the plots of its density, survival and hazard functions, we conclude that this model could be used to analyze skewed data sets with different shapes as well as time or age dependent random variables.

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