Applications of some Exponential Related Distributions

M. A. Umar†1; J. O. Jimoh *; M. K. Garba†; R. B. Afolayan†; W. B. Yahya†5

†Department of Statistics,
University of Ilorin, Ilorin, Nigeria.

*School of Preliminary Studies, Science Department,,
Kogi State Polytechnic,
Lokoja, Nigeria.
E-mail: mumarad90@gmail.com[1]; wbyahya@unilorin.edu.ng[5]

***Abstract* — Exponential family of distribution is a very popular family of distribution functions for analyzing any lifetime data with a lot of applications in different fields of knowledge. This family has distribution functions whose survival, hazard and mean residual life functions are simple and easy to study. As a result, this family has been generalized, modified and mixed with other density functions to give more flexible density functions to facilitate better modeling and analysis. This study is carried out to apply some of these distributions to real life datasets from clinical sciences, remission times of a Bladder cancer dataset. The behaviours of these distributions were illustrated graphically. The parameters of the distributions were estimated using the maximum likelihood method and their goodness-of-fits were examined. The distributions were found to provide satisfactory fits to the datasets considered. Exponentiated Exponential and Exponential-Gamma distributions were found to perform better than all the competing distributions.**

Keywords- *Exponential family, Gamma, Weibull, Lindley, Hazard rate, Mean residual life, Goodness-of-fit.*

I. Introduction

The exponential family of distribution is very popular for analysing any lifetime data. It has a lot of applications in different fields of knowledge. Although exponential distribution is a special case of gamma distribution, the gamma distribution function or survival function cannot be expressed in a closed form. This makes it a little bit unpopular as compared to the Weibull, Exponential, Lindley, to mention but few distributions, whose survival functions and hazard functions are simple and easy to study (Rather & Subramanian, 2019). A generalization of the exponential family was introduced as a new concept of distributions by Gupta et al (1998) and discussed a new family of distributions called the exponentiated exponential distribution. The family has two parameters scale and shape, which are similar to the Weibull or Gamma family. Later Gupta & Kundu (2001) studied some properties of the distribution.

Many properties of the new family were observed to be similar to those of the Weibull or Gamma family. Hence the distribution can be used as alternative to a Weibull or Gamma distribution. The two-parameter Gamma and Weibull are the most popular distributions for analyzing any lifetime data. These parameters, representing the scale and the shape parameter make them quite a bit flexible to analyse any positive real data. As a result, the exponential family has been generalized, re-generalized, modified and mixed with other density functions to give more flexible density functions and facilitate better modeling. This is because the distributions of combined random variables are more flexible, perform better and have wider applicability (Akarawak et al, 2017) according to Ogunwale et al (2019) and as shown by several other studies (Gupta & Kundu, 2001; Nadaraja & Kotz, 2005; Pal et al, 2006). Their mathematical properties are widely studied and used on real life datasets from applied sciences and other fields of knowledge (Abouammoh et al, 2015; Alkarni, 2015; Bhati et al, 2015; Ghitany, 1998; Ghitany et al, 2013; Nadarajah, 2008; Parai et al, 2015; Shanker & Shukla, 2019; Sharma et al, 2015; Wang, 2013; Warahena-Liyanage & Pararai, 2014, among others).

Another member of this family is the three-parameter Generalized Gamma (GG) distribution (Stacy, 1962). The distribution is suitable for modeling data having different types of hazard rate functions; increasing, decreasing, bathtub shaped and unimodal, which makes it particularly useful for estimating individual hazard functions. The distribution has been used in different fields of knowledge such as engineering, hydrology and survival analysis (Shanker & Shukla, 2017). It has Weibull, Gamma and Exponential distributions as special sub-models. It plays a very important role in statistical inferential problems. A generalization of the Generalized Gamma (GGG) distribution, which includes the three-parameter generalized gamma (GG) distribution, two-parameter Weibull and gamma distributions, and exponential distribution, has been suggested and investigated by Shanker & Shukla (2019). The behavior of the hazard rate function of these distributions has been discussed. The estimation of their parameters is explained using the method of maximum likelihood. The goodness-of-fit of these distributions has been discussed using real life datasets from different fields of knowledge.

The analysis and modeling of lifetime data is a great interest of researchers in applied sciences including engineering, medical sciences, insurance, and finance, among others (Shanker, 2016), thus, models are constructed to facilitate better modeling and significant progress (Asad et al, 2018).

A number of lifetime distributions have been proposed and introduced in the literature using members of this family. This because constant hazard rate and mean residual life function is attributed to the exponential distribution whereas the modifications and generalizations made have different shapes of hazard rate and mean residual life functions; that is, increasing, decreasing, bathtub shaped, J-shaped, unimodal, and so on (Abouammoh et al, 2015; Alkarni, 2015; Bhati et al, 2015; Ghitany, 1998; Ghitany et al, 2013; Lindley, 1958; Nadarajah, 2008; Parai et al, 2015; Shanker & Shukla, 2019; Sharma et al, 2015; Wang, 2013; Warahena-Liyanage & Pararai, 2014, among others) which makes them particularly useful for estimating individual hazard functions. The family has been used in the context of Fiducial and Bayesian Statistics (Lindley, 1958), reliability studies (Ghitany et al, 2008) among others. However, this paper is aimed at applying some of these distributions to a real life data from clinical sciences.

II. Research Methodology

The Gamma distribution (Jambunathan, 1954) is defined by its probability density function as;

$$f(x;\alpha,\theta)=\frac{1}{\Gamma(\alpha)}\theta^{\alpha}x^{\alpha-1}e^{-\theta x};x>0,\alpha>0,\theta>0 \quad (1)$$

The distribution in (1) has been generalized to give the Generalized Gamma (GG) distribution (Stacy, 1962) as;

$$f_1(x;\alpha,\theta,\beta)=\frac{1}{\Gamma(\alpha)}\beta\theta^{\alpha}x^{\beta\alpha-1}e^{-\theta x^{\beta}};x>0,\alpha>0,\theta>0,\beta>0 \quad (2)$$

Shanker & Shukla (2019) generalized the GG distribution to give Generalization of the Generalized Gamma (GGG) distribution as;

$$f_2(x;\alpha,\theta,\beta,\gamma)=\frac{1}{\Gamma(\alpha)}\beta\left(\theta\gamma^{\beta}\right)^{\alpha}x^{\beta\alpha-1}e^{-\theta(\gamma x)^{\beta}};x>0,\alpha>0,\theta>0,\beta>0,\gamma>0 \quad (3)$$

The exponential distribution (which is a special case of the gamma) is defined by as;

$$f(x)=\theta e^{-\theta x};x>0,\theta>0 \quad (4)$$

The exponentiated exponential distribution is defined (Gupta & Kundu, 2001) as

$$f(x;\alpha,\theta)=\alpha\theta\left(1-e^{-\theta x}\right)^{\alpha-1}e^{-\theta x};x>0,\alpha>0,\theta>0 \quad (5)$$

The Weibull distribution is defined (Weibull, 1951) as:

$$f(x;\ \alpha,\theta)=\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^{\alpha}},x>0,\alpha>0,\theta>0 \quad (6)$$

The Lindley distribution (Lindley, 1958) is defined as:

$$f(x;\ \theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x};\ x>0,\theta>0 \quad (7)$$

The Exponential-Gamma distribution (Ogunwale et al, 2019) is defined by its p.d.f as:

$$f(x;\alpha,\theta)=\frac{x^{\alpha-1}\theta^{\alpha+1}e^{-2\theta x}}{\Gamma(\alpha)};\ x>0,\alpha>0,\theta>0 \quad (8)$$

The New Exponential-Gamma (NEG) distribution (Umar & Yahya, 2019) is defined by its density function as:

$$f(x;\alpha,\theta)=\frac{\theta}{\theta+\Gamma(\alpha)}\left(\theta+\theta^{\alpha-1}x^{\alpha-1}\right)e^{-\theta x};\ x>0,\alpha>0,\theta>0 \quad (9)$$

The expression in (9) reduces to exponential distribution in (4) when $\alpha=1$ and it reduces to a Lindley distribution in (7) when $\alpha=2$, and also reduces to a number of other exponential family members at different values of the parameter α with different mixing proportions, for example;

it reduces to Shanker, Akash, Rama and Rani distributions when $\alpha = 2, 3, 4$ and 5, respectively (Shanker, 2015a, 2015b, 2017a and 2017b), each having a different mixing proportion. The Lindley distribution has been modified, extended and generalized along with its applications to different fields of knowledge to introduce more flexible density functions such as:

1. The Power Lindley distribution by Ghitany, et al., (2013) as;

$$f(x;\theta,\beta) = \frac{\beta\theta^2}{\theta+1}\left(1+x^{\beta}\right)x^{\beta-1}e^{-\theta x^{\beta}}; x>0, \theta, \beta > 0 \quad (10)$$

2. The Exponentiated Power Lindley distribution by Warahena-Liyanage & Pararai (2014) as;

$$f(x;\theta,S,\beta) = \frac{S\beta\theta^2}{\theta+1}\left(1+x^{\beta}\right)x^{\beta-1}e^{-\theta x^{\beta}}\left(1-\left[1+\frac{\theta x^{\beta}}{\theta+1}\right]e^{-\theta x^{\beta}}\right)^{S-1}; x>0, \theta>0, \beta>0, S>0 \quad (11)$$

3. The Lindley Stretched Exponential (LSE) distribution by Gulshan & Ahmad (2019)

$$f(x;a,b,\theta) = \frac{b\theta^2}{a(\theta+1)}\left(\frac{x}{a}\right)^{b-1}\left\{1+\left(\frac{x}{b}\right)^{b}\right\}e^{-\theta\left(\frac{x}{b}\right)^{b}}; x>0, (a,b,\theta)>0 \quad (12)$$

It can be easily verified that the Generalization of the Generalized Gamma distribution in (3) reduces to Generalized Gamma distribution in (2) when $\gamma = 1$; while it reduces to Gamma distribution (1) when $\gamma = \beta = 1$; it reduces to Weibull distribution (6) when $\gamma = \alpha = 1$; and finally it reduces to exponential distribution (4) when $\gamma = \beta = \alpha = 1$. Similarly, the Exponentiated exponential distribution in (5) and the Weibull distribution in (6) reduces to Exponential distribution in (4). Moreover, the Exponentiated Power Lindley distribution in (11) reduces to the Power Lindley distribution when $S = 1$; while it reduces to Eponentiated Lindley when $\beta = 1$; and finally reduces to Lindley distribution in (7) when $S = \beta = 1$, and the Lindley Stretched Exponential distribution in (11) reduces to Lindley-Exponential distribution (Gulshan & Ahmad, 2019) when $b = 1$; while it reduces to the Lindley distribution in (7) when $\alpha = b = 1$; finally it reduces to Lindley Negative Exponential distribution (Gulshan & Ahmad, 2019) when $\alpha = 1$. The graphs of some of these distributions for different values of the parameters are shown in the Figure 1.

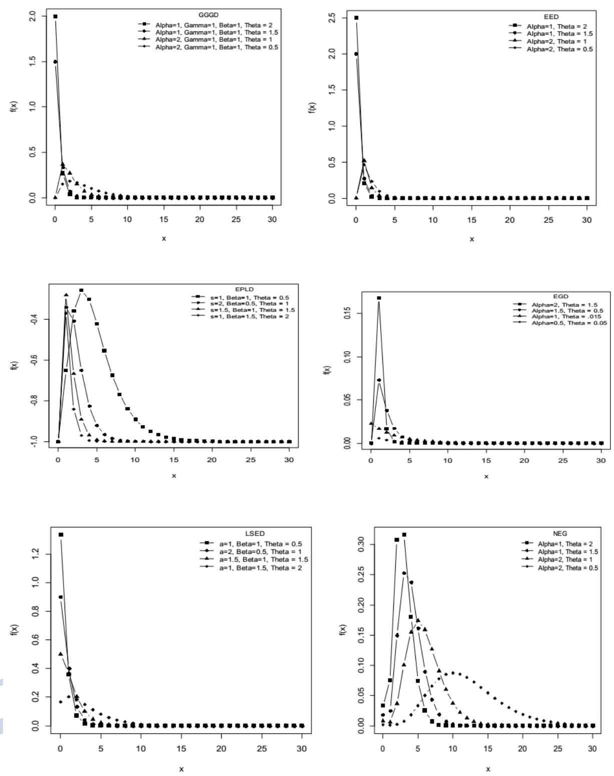

Figure 1: The graph of the distributions at different values of their parameters.

III. Analysis and Results

In this section, the goodness-of-fit of these distributions is discussed with an application to real-life datasets from clinical sciences. The parameters of the distribution were solved using the MLE method while the goodness-of-fit was evaluated using the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) with their respective statistics given as follows:

$$AIC = -2\ln L + 2k \quad (13)$$

$$BIC = -2\ln L + k\ln n \quad (14)$$

Where k is the number of parameters and n is the sample size. The distribution that has a lower value of these criteria is judged to be the best among others

Data Description

The dataset used is an uncensored dataset corresponding to remission times (in months) of a random sample of 118 bladder cancer patients reported in Lee & Wang (2003).

*Table 1: **Parameters Estimation and Goodness-of-Fit Test Results of the distributions for the Bladder cancer dataset.***

Distribution	ML Estimate	-2logLik	AIC	BIC
NEGD	$\hat{\alpha} = 1.4616$ $\hat{\theta} = 0.1041$	742.679	746.679	747.449
EGD	$\hat{\alpha} = 1.9458$ $\hat{\theta} = 0.0015$	731.254	735.254	738.502
LSED	$\hat{\alpha} = 223.5201$ $\hat{\theta} = 1.0473$ $\hat{\beta} = 28.0779$	742.235	748.235	747.006
EPLD	$\hat{\theta} = 0.7914$ $\hat{\beta} = 0.5566$ $\hat{S} = 2.0433$	739.368	745.368	744.139
GGGD	$\hat{\alpha} = 2.3479$ $\hat{\theta} = 0.8246$ $\hat{\beta} = 0.5744$ $\hat{\gamma} = 0.9112$	739.584	747.584	744.355
GGD	$\hat{\alpha} = 2.3459$ $\hat{\theta} = 0.7809$ $\hat{\beta} = 0.5747$	739.584	747.584	744.355
GD	$\hat{\alpha} = 0.3856$ $\hat{\theta} = 0.7437$	742.234	746.234	747.005
WD	$\hat{\alpha} = 0.1046$ $\hat{\theta} = 1.0478$	828.174	832.174	837.878
LD	$\hat{\theta} = 0.1961$	839.060	841.060	843.912
EED	$\hat{\alpha} = 0.9296$ $\hat{\theta} = 0.1108$	726.695	730.695	745.778
ED	$\hat{\theta} = 0.1163$	743.718	745.718	748.489

IV. Concluding Remarks

The exponential related distributions were implemented on the Bladder cancer dataset to examine their goodness-of-fit while their efficiency was examined and compared among the distributions. The Exponentiated Exponential (EE) distribution (Gupta & Kundu, 2001) has the smallest values of AIC and BIC and appreciable results in other performance measures followed by the Exponential-Gamma (EG) distribution (Ogunwale et al, 2019), these were found to perform better than all the distributions considered. Following the results of the goodness-of-fit of Lindley Stretched Exponential (LSE) distribution fitted as reported in Gulshan & Ahmad (2019), it is obvious that these distributions performed better.

References

Abouammoh, A. M., Alshangiti, A. M. and Ragab, I. E. (2015). A new generalized Lindley distribution. *Journal of Statistical Computation and Simulation*. Preprint http://dx.doi.org/10.1080/00949655.2014.995101.

Akarawak, E. E. E., Adeleke, I. A. & Okafor, R. O. (2017), The Gamma-Rayleigh Distribution and Applications to Survival Data. *Nigerian Journal of Basic and Applied Science, **25***(2): 130-142

Alkarni, S. (2015). Extended power Lindley distribution - a new statistical model for non-monotone survival data. *European journal of statistics and probability*. *3*(3), pp. 19–34.

Asad, A., Qaisar, R., Muhammad, Z. and Muhammad, T. J. (2018). A Quasi Lindley Pareto distribution. *Proceedings of the Pakistan Academy of Sciences: A Physical and Computational Science. **55***(2): 32 – 40.

Bhati, D., Malik, M. A. And Vaman, H. J. (2015). Lindley-exponential distribution: Properties and applications. *Metron, 73*(3), pp. 335-357.

Ghitany, M. E., Atieh, B. and Nadarajah, S. (2008). Poisson-Lindley distribution and Its Applications, *Mathematics Computation and Simulation*, ***78***: 493-506.

Ghitany, M., Al-Mutairi, D., Balakrishnan, N. and Al-Enezi, I. (2013). Power Lindley distribution and associated inference. *Computational Statistics and Data Analysis*, 64, pp. 20-33.

Ghitany, M.E. (1998). On the recent Generalization of Gamma Distribution. *Communication in Statistics-Theory and Methods*, 27(1); 223 – 233.

Gulshan, A. M. and Ahamd S. A. (2019). Lindley Stretched Exponential distribution with Applications. *Global Scientific Journals. 7*(1). ISSN 2320 – 9186.

Gupta, R. D. & Kundu, D. (2001). Exponentiated Exponential Family: An Alternative to Gamma and Weibull Distributions. *Biometrical Journal, **43***(1): 319 – 326.

Jambunathan, M. V. (1954). Some properties of beta and gamma distributions. *Annals of Mathematical Statistics, **25***; 401-405.

Lee, E. T. and Wang, J. W. (2003). *Statistical methods for survival data analysis, 3rd edition*, Wiley, New York.

Lindley, D. V. (1958). Fiducial distributions and Bayes' theorem. *Journal of the Royal Statistical Society, Series B, **20***. Pp. 102–107.

Nadaraja, S. & Kotz, S. (2005). The Beta-Exponential distribution. *Reliability Engineering and System Safety **91***; 689 – 697

Nadarajah, S. (2008). Some Gamma Distributions, Statistics: *A Journal of Theoretical and Applied Statistics*, ***42***(1), 77-94.

Ogunwale, O. D., Adewusi, O. A. & Ayeni T. M. (2019). Exponential-Gamma Distribution. *International Journal of Emerging Technology and Advanced Engineering, **9***(10); 245-249.

Pal, M., Ali, M. M. & Woo, J. (2006). Exponentiated Weibull Distribution. *Statistica, **66***(2): 139-148.

Pararai, M., Liyanage, G. W. and Oluyede, B. O. (2015). A new class of generalized Power Lindley distribution with applications to lifetime data. *Theoretical Mathematics & Applications*, ***5***(1), pp. 53–96.

R Core Team (2018). R: A Language and Environment for Statistical Computing. R Foundation for Statistical Computing, Vienna, Austria. http://www.R-project.org.

Rather, A. A. and Subramanian, C. (2019). ExponentiatedIshita Distribution: Properties andApplications. *International Journal of Management, Technology and Engineering, **9***(5); 2473 – 2484.

Shanker, R. (2015a). Shanker Distribution and Its Applications. *International Journal of Statistics and Applications*. ***5***(6): 338 -348.

Shanker, R. (2015b). Akash Distribution and Its Applications. *International Journal of Probability and Statistics*. ***4***(3): 65-75.

Shanker, R. (2016). Sujatha Distribution and its Applications. *Statistics in Transition New series*; ***17***(3): 391–410.

Shanker, R. (2017a). Rama Distribution and Its Application. *International Journal of Statistics and Applications*, *7*(1): 26-35. http://DOI:10.5923/j.statistics.20170701.04

Shanker, R. (2017b). Rani Distribution and Its Application. *Biometrics & Biostatistics International Journal*. ***6***(1): 1-20.

Shanker, R. and Shukla, K. K. (2017). On modeling lifetime data using three parameter generalized Lindley and generalized gamma distribution. *Biometrics & Biostatistics International Journal, **4***(7), 1 – 7.

Shanker, R. and Shukla, K. K. (2019).A Generalization of Generalized Gamma Distribution.*International*

Journal of Computational and Theoretical Statistics, ***6***(1); 33 – 42. http://dx.doi.org/10.12785/ijcts/060105

Sharma, V., Singh, S., Singh, U. and Agiwal, V. (2015). The inverse Lindley distribution - a stress-strength reliability model with applications to head and neck cancer data. *Journal of Industrial & Production Engineering*, ***32*** (3), pp. 162–173.

Stacy, E. W. (1962). A generalization of the Gamma Distribution. *Annals of Mathematical Statistics,* ***33***, 1187 – 1192.

Wang, M. (2013). A new three-parameter lifetime distribution and associated inference, *Computational Statistics and Data Analysis*, ***32***(6), pp. 120-133.

Warahena-Liyanage, G. and Perarai, M. (2014). A Generalized Power Lindley Distribution with Applications. *Asian Journal of Mathematics and Applications (article ID ama0169)*; 1 – 23.

Weibull W. (1951). A statistical distribution function of wide applicability. *J Appl Mech - Trans ASME;* ***18***(3):293–7.

.