Type II Half Logistic Exponentiated Exponential Distribution: Properties and Applications

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***Abstract* — This paper introduced a new distribution called Type II Half Logistic Exponentiated Exponential (TIIHLEE) distribution established from Type II half logistic G family of distribution. Some mathematical properties; moments, probability weighted moments, mean deviation, quantile function, Renyi entropy of TIIHLEE distribution are investigated. The expressions of order statistics are derived. Parameters of the derived distribution are obtained using the maximum likelihood method. We compare the fits of the TIIHLEE model with some models which shows that the new model best fitted the data than other models.**

Keywords - *Type II half logistic-G family, Moments, Order statistic, Estimation, Maximum likelihood, quantile function.*

I. Introduction

Since real-world data are usually complex and can take a variety of shapes, existing distributions do not always provide an adequate fit. Hence, generalizing distributions and studying their flexibility is of interest to researchers for the last decades, the generated family of continuous distributions is a new improvement for producing and extending the usual classical distributions. The half logistic distribution is a member of the family of logistic distributions which is introduced by [1] which has the following cumulative distribution function (cdf)

$$F(t) = \frac{1-e^{\lambda t}}{1+e^{\lambda t}} \qquad t>0, \quad \lambda>0 \tag{1}$$

The associated probability density function (pdf) corresponding

$$f(t) = \frac{2e^{-\lambda t}}{(1+e^{\lambda t})^2} \qquad t>0, \quad \lambda>0 \tag{2}$$

[2] use the half logistic generator instead of gamma generator to obtain type II half logistic family which is denoted by TIIHL-*G*.
Then the cdf and pdf of TIIHL-G are define as follows

$$F(x;\lambda) = 1 - \int_0^{-\log G(x)} \frac{2\lambda e^{-\lambda}}{\left(1+e^{-\lambda t}\right)^2} dt = \frac{2[G(x)]^{\lambda}}{1+[G(x)]^{\lambda}} \qquad x>0, \lambda>0 \tag{3}$$

$$f(x;\lambda) = \frac{2\lambda g(x)[G(x)]^{\lambda-1}}{\left[1+[G(x)]^{\lambda}\right]^2} \qquad x>0, \lambda>0 \tag{4}$$

Where λ is the shape parameter and g(x;λ) and G(x;λ) is the baseline distribution respectively.
However, the TIIHLEE has tractable properties especially for simulation since its quantile function takes a simple form.

$$Q(u) = G^{-1}\left[\frac{u}{2-u}\right]^{\frac{1}{\lambda}} \tag{5}$$

where u is a uniform distribution on the interval (0,1) and G^{-1} (.) is the inverse function of *G*(.)

Our aim in this work is to study a modified statistical distribution that will be suitable to fit positively skewed and unimodal data and to check the flexibility of the existing and proposed distribution.

II. Research Methodology

A. Type II Half Logistic Exponentiated Exponential Distribution

The cumulative density function (cdf) and probability density function (pdf) of exponentiated exponential distribution (EE) are define as follows:

$$G(x;\alpha,\beta) = (1-e^{-\beta x})^{\alpha} \tag{6}$$
$$g(x;\alpha,\beta) = \alpha\beta e^{-\beta x}(1-e^{-\beta x})^{\alpha-1} \tag{7}$$

Substituting 6 and 7 in 3 and 4 then we define probability density function (pdf) and cumulative density function (cdf) Type II Half Logistic Exponentiated Exponential Distribution (TIIHL-EE) as follows:

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$$F(x;\alpha,\beta,\lambda)=\frac{2\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda}}{1+\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda}} \quad (8)$$

$$f(x;\alpha,\beta,\lambda)=\frac{2\lambda\alpha\beta e^{-\beta x}\left(1-e^{-\beta x}\right)^{\alpha-1}\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda-1}}{\left[1+\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda}\right]^{2}} \quad (9)$$

Henceforth, a random variable with probability density function (pdf) is denoted by $X \sim TIIHLEE\ (\alpha,\beta,\lambda)$

B. Quantile function

$$x=-\frac{1}{\beta}In\left(1-u^{\frac{1}{\alpha}}\right)\left(\frac{u}{2-u}\right)^{\frac{1}{\lambda}} \quad (10)$$

where α *and* λ are shape parameters while β is a scale parameter.

C. Moment of Proposed TIIHLEE Distribution

The r[th] moment for the TIIIHL-*G* family is derived.

$$\mu^r=\int_0^\infty x^r f(x)dx$$

$$\mu^r=\alpha\beta(i+1)\int x^r e^{-\beta x}\left(1-e^{-\beta x}\right)^{\alpha(i+1)-1}dx \quad (11)$$

However, the equation (11) can be written as

$$\mu^r=\frac{\Gamma\left(\frac{r}{2}+1\right)}{\alpha\beta^{r+2}(k+1)^{\frac{r}{2}+1}}w_{i.j,k} \quad (12)$$

D. Moment Generating Function of TIIHLEE Distribution

A random variable x with pdf f(x) is defined a

$$M_x(t)=E(e^{tx})$$

$$E(e^{tx})=\int_0^\infty e^{tx}f(x)dx$$

$$=\sum_{r=0}^\infty \frac{t^r}{r!}\mu_r^1(x)$$

$$=\sum_{r=0}^\infty \frac{t^r}{r!}E(X^r)$$

The mean of the proposed TIIHLEE distribution is gotten by making μ^r moment equal to one (|r=1)

$$\mu_j^1=\frac{\Gamma\left(\frac{3}{2}\right)}{\alpha\beta^3(k+1)^{\frac{3}{2}}}w_{i.j,k} \quad (13)$$

When r=2

$$\mu_j^2=\frac{\Gamma(2)}{\alpha\beta^4(k+1)^2}w_{i.j,k} \quad (14)$$

When r=3

$$\mu_j^3=\frac{\Gamma\left(\frac{5}{2}\right)}{\alpha\beta^5(k+1)^{\frac{5}{2}}}w_{i.j,k} \quad (15)$$

$$\mu_j^4=\frac{\Gamma(3)}{\alpha\beta^6(k+1)^3}w_{i.j,k} \quad (16)$$

The variance of TIIHLEE can be obtain as

$$\mathrm{var}(x)=\sigma^2=E(x^2)-[E(x)]^2=\mu_j^2-\left(\mu_j^1\right)^2 \quad (17)$$

Substituting equation 14 and 13 into 17 to obtain the variance of TIIHLEE

$$\sigma^2=\frac{\Gamma(2)}{\alpha\beta^4(k+1)^2}w_{i.j,k}-\left(\frac{\Gamma\left(\frac{3}{2}\right)}{\alpha\beta^3(k+1)^{\frac{3}{2}}}w_{i.j,k}\right)^2$$

$$\sigma^2=\frac{\Gamma(2)}{\alpha\beta^4(k+1)^2}w_{i.j,k}-\frac{\left(\Gamma\left(\frac{3}{2}\right)\right)^2}{\alpha\beta^6(k+1)^{\frac{9}{4}}}w_{i.j,k} \quad (18)$$

$$=\int_0^\infty e^{tx}\alpha\beta e^{-\beta x}(i+1)\left(1-e^{-\beta x}\right)^{\alpha(i+1)-1}dx$$

$$=\alpha\beta\int_0^\infty e^{(t-\beta)x}(i+1)\left(1-e^{-\beta x}\right)^{\alpha(i+1)-1}dx$$

$$=\sum_{i,j,k=0}^\infty \frac{t^r\Gamma\left(\frac{r}{2}+1\right)}{r!\alpha\beta^{r+2}(k+1)^{\frac{r}{2}+1}}w_{i,j,k} \quad (19)$$

Obtaining the first moment from the moment generating function we differentiate with respect to

$$\mu_j^1 = M_x^1(t)$$

$$M_x^1(t) = \frac{rt^{r-1}}{r!}\sum_{i,j,k=0}^{\infty}\frac{\Gamma\left(\frac{r}{2}+1\right)}{r!\alpha\beta^{r+2}(k+1)^{\frac{r}{2}+1}}w_{i,j,k}$$

$$M_x^1(0) = \sum_{i,j,k=0}^{\infty}\frac{\Gamma\left(\frac{r}{2}+1\right)}{r!\alpha\beta^{r+2}(k+1)^{\frac{r}{2}+1}}w_{i,j,k}$$

E. Order Statistic of TIIHLEE

$$f(x;\alpha,\beta,\lambda) = \frac{f(x;\alpha,\beta,\lambda)}{B(r,n-r-1)}\sum_{v=0}^{n-r}(-1)^v\binom{n-r}{v}F(x;\alpha,\beta,\lambda)^{v+r-1} \tag{21}$$

$B(.,.)$is the beta function. The pdf of the r[th] order statistic for TIIHLEE distribution is derived by substituting (8) and (9) in (21) as follows

$$f(x;\alpha,\beta,\lambda) = \frac{1}{B(r,n-r+1)}\sum_{v=0}^{n-r}(-1)^v\binom{n-r}{v}\frac{\alpha\beta\lambda e^{-\beta x}\left(1-e^{-\beta x}\right)^{\alpha-1}\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda(v+r)-1}}{\left[1+\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda+r+1}} \tag{22}$$

Applying the binomial expansion in (40), then we have

$$f(x;\alpha,\beta,\lambda) = \frac{1}{B(r,n-r+1)}\sum_{v=0}^{n-r}(-1)^{v+i}\binom{n-r}{v}\binom{v+r+i}{i}\alpha\beta\lambda e^{-\beta x}\left(1-e^{\beta x}\right)^{\alpha-1}\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda(v+r+i)-1}$$

Again, using the binomial expansion in the previous equation, then the pdf of the r[th] order statistic for TIIHLEE distribution is obtained as follows

$$f(x;\alpha,\beta,\lambda) = \frac{1}{B(r,n-r+1)}\sum_{v=0}^{n-r}\sum_{I=0}^{v+r-1}\sum_{J=0}^{\infty}\eta^{*}\alpha\beta\lambda e^{-\beta(j+1)x^2} \tag{23}$$

$$\eta^{*} = (-1)^{v+i+j}\binom{n-r}{v}\binom{v+r+i}{i}\binom{\lambda(v+r+i)-1}{j}$$

F. Maximum Likelihood Estimator (MLE)

Let $x_1; x_2;\ldots, x_n$ be a random sample of size n from the TIIHL family of distributions ($\alpha; \beta; \lambda$). The log-likelihood function for the vector of parameters L = $(\alpha; \beta; \lambda)^T$ can be expressed as

$$\text{Let } f\left(x_1,x_2,x_3,\ldots,x_n;\alpha,\beta,\lambda\right) = \prod_{i=1}^{n} f(x)$$

$$\prod_{i=1}^{n}\left[\frac{2\lambda\alpha\beta e^{-\beta x}\left(1-e^{-\beta x}\right)^{\alpha-1}\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda-1}}{\left[1+\left[\left(1+e^{-\beta x}\right)^{\alpha}\right]^{\lambda}\right]^{2}}\right]$$

The log-likelihood function is expressed as

$$l = n\log 2 + n\log\lambda + n\log\alpha + n\log\beta + \alpha - 1\sum_{i=1}^{n}\log\left(1-e^{-\beta x}\right) + (\lambda-1)\sum_{i=1}^{n}\log\left(1-e^{-\beta x}\right)^{\alpha}$$

$$-2\sum_{i=1}^{n}\log\left[1+\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda}\right] \tag{24}$$

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Taking the first partial derivatives of $\ell(x; \alpha, \beta, \lambda)$ of (24) with respect to α, β, and λ and letting them equal zero, we obtain a nonlinear system of equations.

$$\frac{\delta l}{\delta \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log\left(1 - e^{-\beta x}\right) - \frac{(\lambda - 1)(\alpha - 1)\beta \sum_{i=1}^{n} x\left(1 - e^{-\beta x}\right)^{\alpha - 1}}{\left(1 - e^{-\beta x}\right)} - 2\sum_{i=1}^{n} \frac{\left[\left(1 - e^{-\beta x}\right)\right]^{\lambda} \log\left[\left(1 - e^{-\beta x}\right)^{\alpha}\right]}{\left[1 + \left(1 - e^{-\beta x}\right)^{\alpha}\right]^{\lambda}} = 0$$

$$\frac{\delta l}{\delta \beta} = \frac{n}{\beta} + \frac{(\alpha - 1)\sum_{i=1}^{n} \beta e^{\beta x}}{\left(1 - e^{-\beta x}\right)} - (\lambda - 1)\beta \sum_{i=1}^{n} x\left(1 - e^{-\beta x}\right) - 2\sum_{i=1}^{n} \frac{\left[\left(1 - e^{-\beta x}\right)^{\alpha}\right]^{\lambda - 1} x e^{-\beta x}}{\left[1 + \left[(1 - e^{-\beta x})^{\alpha}\right]^{\lambda}\right]} = 0$$

$$\frac{\delta l}{\delta \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} \log\left(1 - e^{-\beta x}\right) - 2\sum_{i=1}^{n} \frac{\left[\left(1 - e^{-\beta x}\right)^{\alpha}\right]^{\lambda} \log\left[\left(1 - e^{-\beta x}\right)^{\alpha}\right]}{\left[1 + \left(1 - e^{-\beta x}\right)^{\alpha}\right]^{\lambda}} = 0$$

III. Results And Discussion

A. Results

To illustrate the importance and flexibility of the TIIHLEE distribution, two real data sets are demonstrated. We compare the fits of the TIIHLEE model with some models namely; Type II half logistic exponential (TIIHLE) by [3], Half logistic Exponentiated Exponential (HLEE), Half Logistic Exponential (HLE), by [4] Exponential distribution (E) by [5], Exponentiated Exponential (EE) by [6], Marshal Olkin extended exponential (MOEE) by [7] and Weibull distribution.

Tables 1 and 2 gives the MLEs of the unknown param eter(s), the goodness of fit measures log-likelihood (*L L*), Akaike Information Criterion (AIC), Bayesian Inf ormation Criterion (BIC), Anderson-Darling (A^*), Cra mér-von Mises (W^*) and Kolmogorov-Smirnov (K-S) for the fitted lifetime distribution.

The first data on the remission times (in months) of a random sample of 128 bladder cancer patients studied by [8] reported by [9] and [10].

B. Discussion

The TIIHLEE distribution provides a better fit to the bladder cancer patient data than other competitive models, the TIIHLE, EE, E and the W distribution.

However, from Table 1, the TIIHLEE distribution has the highest log-likelihood and the smallest K-S, W^*, A^*, AIC, and BIC values compared to the other models. Although the TIIHLEE distribution provides the best fit to the data, the EE and W distribution are alternatively good models for the data since their measures of fit values are close to that of the TIIHLEE distribution. Fig 1 shows the histogram of the bladder cancer data set along with the fitted model. The theoretical density of TIIHLEE distribution has a better spread to the right than other models on the data set.

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Table 1: Maximum Likelihood Estimates of Parameters, their Standard Errors, Goodness-of-fit Statistics, Log-likelihood and Information Criteria for Bladder Cancer Patient

	Estimated Parameter								
Distributions	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	**AIC**	**BIC**	**LL**	**A***	**W***	**K-S**
TIIHLEE	2.399 (9.0337)	0.0979 (0.0121)	0.6235 (2.4072)	**828.0746**	**836.6307**	**-411.0373**	**0.33438**	**0.05441**	**0.05088**
TIIHLE	1.4586 (0.1572)	0.0978 (0.0121)		1003.520	1009.224	-449.7602	0.33622	0.05482	0.05482
EE	1.2186 (0.1489)	8.2466 (0.9228)		830.1552	835.8593	-413.0776	0.7110	0.1272	0.0724
E	6.0891 (0.3998)			1023.348	1026.200	-510.6739	11.2831	1.9939	0.1792
W	0.0939 (0.0191)	1.0477 (0.6750)		832.1738	837.8777 8	-414.0869	0.9593	0.1541	0.0701S

Source: [8]

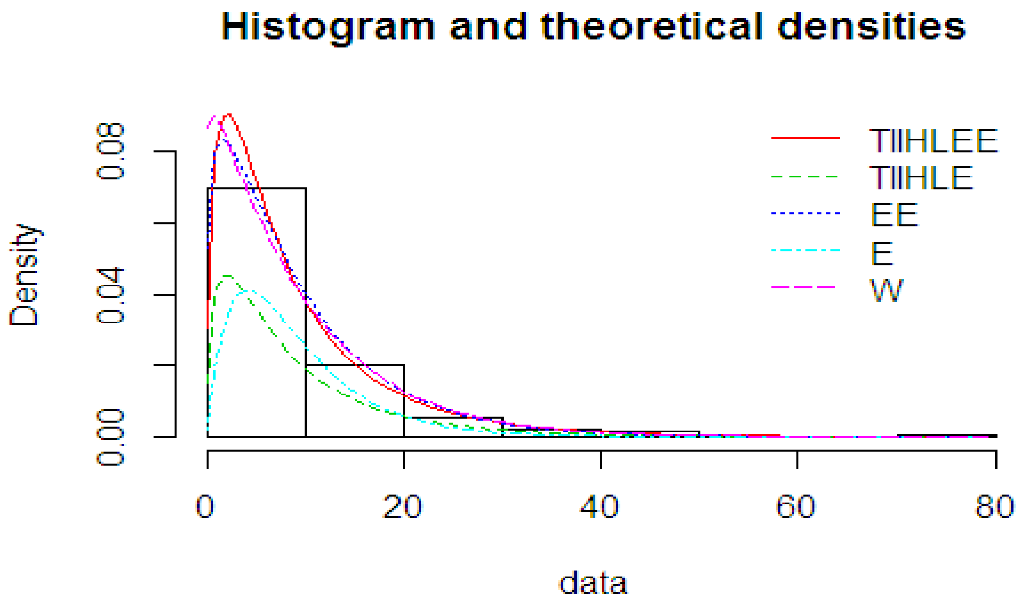

Fig 1: Goodness-of-fit Plot of TIIHLEE Distribution with other related distributions fitted to Bladder Cancer Data.

IV. Conclusion

In this study, we proposed a modified two-parameter which is added to the type II half logistic family of distribution, called Type II Half Logistic Exponentiated Exponential (TIIHLEE). Some structural mathematical properties; Moment Incomplete moments, Probability Weighted Moment, Order Statistic, and Rényi entropy of the derived model are investigated. A simulation study is carried out to estimate the behavior of the shape and scale model parameters, also maximum likelihood estimators were investigator. The application of two real-life data set shows that the TIIHLEE strong and better fit than Type II half logistic exponential (TIIHLE), Exponential distribution (E), Exponentiated Exponential (EE), and Weibull distribution. However, the proposed distribution TIIHLEE will serve as an alternative model in the area of theoretical and applied statistics.

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