A Multi-Item Economic Order Quantity Model with Discrete Probabilistic Lead Time Demand

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***Abstract* — In this paper, a multi-item probabilistic inventory model with discrete and probabilistic demand in a fixed reorder and zero lead-time is presented. The model was derived from the EOQ individual and joint replenishment model with Gamma lead time demand. The Gamma demand function of the model was replaced with demand function of unknown distribution which is determined by fitting the demand for both individual and joint product replenishment. The model gives the optimal ordering quantity that minimizes the total cost for each product (individual replenishment policy) as well as the optimal ordering quantity for all products (joint replenishment policy). The total costs of the model consists of purchasing, ordering, handling as well as shortage costs. The model was illustrated with data collected from Winjec Services, Enugu state. The products of the inventory and distribution firm includes 7up, Mirinda, Pepesi, soda water and sprite non-alcoholic drinks. The distribution of the lead time monthly demand of each product for the past five years was fitted to determine its distribution. It was observed that demand for some of the products followed Poisson distribution while some others followed Geometric distribution. The individual replenishment policy as well as the joint replenishment policy were obtained for the products. Comparison of the optimal solution of both policies were carried out. The optimal solution of the joint replenishment policy presents a lower total cost than that of the individual replenishment policy for the given demand distributions.**

Keywords: *Economic order quantity, multi-item inventory, probabilistic demand, inventory and ordering, replenishment policy.*

I. Introduction

Inventory control helps management of companies to improve the productivity of capital by reducing the material costs. It also prevents the large amount of capital being locked up for long periods thereby causing unavailability of funds for other sectors of the company and improving the capital turn-over ratio.

A lot of inventory models have been developed over the last few decades among which is the Economic Order Quantity (EOQ) model introduced by [1]. The simple EOQ concept is that management is confronted with a ordering and inventory holding costs and tries to minimize the total annual costs of carrying inventory and cost of ordering under the condition of known demand. The EOQ model has been the basis for the development of more advanced and complex inventory models.

Several factors have been incorporated in the model like multiple items, quantity discount, deterioration rate, probabilistic models in terms of demand distribution and so on. Also the cost structure of the EOQ model has expanded from the ordering and holding costs to involve additional costs like purchasing and shortage costs. In the EOQ model there is also the issue of when to order and what quantity of products to order to attain the optimal replenishment policy. Companies sometimes order a group of items simultaneously (joint replenishment) while at some other cases the items are ordered individually (individual replenishment).

In the joint replenishment models the several items are ordered from a single supplier and therefore share the costs involved while for the case of individual replenishment the costs are separate for the different items. There have been situations where the joint and/or individual replenishment policy are considered.

In [2] a mathematical model for an inventory problem consisting of several products where lead time demand follows gamma distribution was developed. The objective was to determine the optimal ordering quantity that minimizes the total cost for each product (individual replenishment policy) and also the optimal ordering time for all the products (joint replenishment policy). Also in [3] the separate works on substitution between items and joint replenishment was combined to propose a joint replenishment model with substitution for two products within the framework of the classical EOQ model. The optimal order quantity for each product was obtained while taking into consideration substitution between them so that demand is partially met. Meanwhile an EOQ model with substitution between products and a dynamic inventory replenishment policy presented in [4]. The main purpose of the model was to enable inventory managers to develop ordering policies that ensures that in the event that a specific product runs out and cannot be replenish due to unforeseen circumstances, the consequent increase in demand for related products will not cause further stock out incidents.

In [5] a new general probabilistic multi-item single source inventory model with mixture shortage cost under two restrictions was proposed. One of the restrictions was on the expected varying backorder and the other on the expected varying lost sales cost. Furthermore in [6] an alternative proof for optimality in the solution of the deterministic Economic production quantity (EPQ) with partial backordering was presented. The model determined how much to purchase (order quantity) and when to place the order (the reorder point) by relaxing the assumption that shortages are not allowed and allowing stock out with partial backordering.

In [7] a variation of the economic order quantity (EOQ) model where commutative holding cost is a non-linear function of time was considered. The model was an approximation of the optimal order quantity for perishable goods. Meanwhile a deteriorated multi-item order quantity (EOQ) problem was presented in [8]. They explored the optimal policy for the inventory problem by analyzing the structural properties of the model and introduced a simple model for solving the optimal solution to the problem. In [9] on the other hand the application of inventory model in determining stock control in an organization was investigated.

In the works reviewed so far we came across a multi-item inventory model with Gamma lead time in [2]. In this paper we re-present the model in [2] but with lead time demand following an unknown distribution. So the lead time demand of the products is fitted during application of the model to determine the lead time demand distributions.

The objective remains to determine the optimal ordering quantity that minimizes the total cost for each product (individual replenishment policy) and also for all the products (joint replenishment policy). The model presented is illustrated with data collected from an ordering/inventory firm Winjen Services which is located in Enugu. The total costs of the model includes purchasing, ordering, handling and shortage costs.

II. Research Methodology

A. *The multi-item probabilistic inventory model.*

The probabilistic EOQ model for individual and joint replenishment policies as presented by [2] are given as follows.

i. Individual Replenishment Policy (IRP): The annual total replenishment inventory model with demand during lead time following a gamma distribution is given as;

$$TAC(Q,R) = PD + \frac{SD}{Q} + Ph\left[\frac{Q}{2} + R - E(X)\right] + \frac{\pi D}{Q}\left[\int_R^\infty (x-R)f(x)dx\ \right] \qquad (1)$$

Where

D = Average demand within one planning period

P= Purchase cost per unit

Q=Optimal order quantity

S= Ordering cost per order

h = Holding cost fraction per unit per planning period

π = Shortage cost per unit

$f(x) = \frac{1}{\beta^\alpha \Gamma_\alpha} x^{\alpha-1} e^{\frac{-x}{\beta}}$ is Gamma lead time demand density function

T A C = Total inventory cost

The total inventory cost considered in building the model consists of purchasing cost, ordering cost, holding cost, and shortage cost which are represented as follows;

Purchase cost is expressed as PD

Ordering cost: If the ordering cost is S, then the annual ordering is $\frac{SD}{Q}$ (2)

Holding cost: Given by Ph $(\frac{Q}{2}$+R-E(X)) , (3)

where X is the random variables for demand during lead time.

Shortage cost: The amount of the annual shortage cost can begotten as

$$\frac{\pi D}{Q}\left[\int_R^\infty (x-R)f(x)dx\right. \qquad (4)$$

$$\text{With } Q = \sqrt{\frac{2DS}{Ph}} \qquad (5)$$

The procedure to obtain the optimal order quantity and reorder point for the model follows the algorithm by [10] as follows;

(1) First we start with the EOQ that is Q = $\sqrt{\frac{2DS}{Ph}}$ (6)
(2) Calculate the value of R.
(3) Calculate TAC using equation (1)

ii. Joint Replenishment Policy (JRP):

In the joint replenishment model all items are ordered at the same time. The decision variable is the optimal time of replenishment with a fixed ordering time interval but varying ordering amount for each item. The decision variable is the optimal time of replenishment, T. The relation $Q_j = D_i T$ is used for each item i in the model for the individual replenishment policy and to find the optimal policy T.

The joint replenishment policy model is expressed as;

$$TAC(T) = \frac{S^*}{T} + \left\{\sum_i^n P_i D_i + P_i h_i \left[\frac{TD_j}{2} + R_i - E(X_i)\right] + \frac{\pi_i}{T}\left(\left[\int_R^\infty (x_i - R_i) f(x_i) dx_i\right]\right)\right\} \quad (7)$$

Where n is the number of times the items are jointly ordered.

B. ***Multi-item model with probabilistic lead time demand***

In this work we present and apply the multi-item ordering/inventory model in [2] but rather with the assumption that lead time demand is probabilistic with unknown distribution *f(x)*. The distribution of the lead time demand for each product i will be therefore be determined by fitting a distribution to the demand.

Hence shortage cost is $\frac{\pi D}{Q}\left[\int_R^\infty (x - R) f(x) dx\right]$ (8)

Then taking $\int_R^\infty (x - R) f(x) dx]$

$$= \int_{k=0}^\infty x f(x) - R\int_{k=0}^\infty f(x) \quad (9)$$

Since $\int_{k=0}^\infty f(x) = 1$, We have $E(x) - R$ (10)

Hence shortage cost $= \frac{\pi D}{Q}[E(x) - R]$ (11)

Hence for IRP, $TAC(Q,R) = PD + \frac{SD}{Q} + Ph\left[\frac{Q}{2} + R - E(x)\right] + \frac{\pi D}{Q}[E(x) - R]$ (12)

With

$P_i D_i$ = the company's purchase cost per demand D_i (13)

$\frac{S_i D_i}{Q_i}$ = The ordering cost for the company products (14)

$P_i h_i \left[\frac{Q_i}{2} + R_i - E(x)\right]$ = The holding cost (15)

$\frac{\pi_i D_i}{Q_i}[E(x) - R]$ = The shortage cost (16)

With $E(x)$ as expected demand of individual item during lead time with distribution $f(x)$.

And for Joint Replenishment Policy,

$$\text{TAC (T)} = \frac{\boldsymbol{S^*}}{\boldsymbol{T}} + \sum_{\boldsymbol{i}}^{\boldsymbol{n}}\left\{P_i D_i + \left(P_i h_i\left(\frac{\boldsymbol{TD_i}}{\boldsymbol{2}} + R_i - E(x_i)\right)\right) + \frac{\boldsymbol{\pi_i}}{\boldsymbol{T}}(E(x_i) - R_i)\right\}$$

where $T = \sqrt{\frac{2(S^* + \sum_{i=1}^n (\pi(E(x_{i)} - R_i)}{\sum_{i=1}^n P_i D_i h_i}}$

With $\sum_{i=1}^n \{E(x_i)\}$ as joint expected demand of i items during lead time.

S* = joint ordering cost $< \sum_{i=1}^n S_i$ and S* = α $\sum_{i=1}^n S_i$

α being a discount factor.

$Q_i = D_i T$

Reorder point for each product R_i given as

$$R_i = \frac{\mathrm{D}_i}{\text{Working days per annum}} \times \mathrm{t}$$

Where

D = Average demand within one planning period
P = Purchase cost per unit
Q = Optimal order quantity
S = Ordering cost per order
S* = ordering cost for joint replenishment policy
h = Holding cost fraction per unit per planning period
π = Shortage cost per unit
T = optimal replenishment time
t = lead time

Assumptions of the model

(1) Demands during lead time is discrete and probabilistic with distribution *f(x)*.
(2) Storage facility is unlimited.
(3) Lead time is known and constant.
(4) Shortage occurs when demands exceeds the inventory level during lead time
(5) The optimal ordering quantity varies in each replenishment

Holding cost fraction depends on the average goods that are stored and remains constant for both replenishment policies.

III. Results And Discussion

A. Data Illustration

The individual and joint replenishment models with unkown discrete probabilistic lead time demand is illustrated with data collected from Winjen services which is an ordering and inventory holding firm. The data used in this work was collected from the company's purchase and ordering unit. The data includes the average annual demand and costs. The costs considered are annual purchase cost , annual ordering cost, annual shortage cost , annual holding fraction for the non-alcoholic drinks 7up, Mirinda, Pepsi, Sprite, and Soda water (in crates) for the year 2018. This is shown in table 1. The demand for the products were each fitted to determine their distributions using the EasyFit software. The Anderson Darling test was used to fit the distribution of the demands for the products from data of previous demands of each product. The resulting distribution for each drink is presented in table 2 together with the parameters of the distribution of lead time demand of the different products.

Table 1. Annual demand and different costs of the drinks.

	7up drink	Mirinda drink	Pepsi drink	Soda water drink	Sprite drink
Annual demand (crate of drink)	1691	1762	1649	1245	1488
Ordering cost	66760	70480	65960	49420	59520
Holding cost fraction	0.21	0.25	0.22	0.17	0.16
Shortage cost	5950	5950	7000	4600	5750
Purchase cost	1135.04	1275.20	1199.73	1150	1149.97

The parameter λ for the Poisson distribution represent the demand in lead time while P for the geometric distribution represents the probability the reorder quantity meets the demand at end of the lead time.

Meanwhile the expected demand of the drinks for the Poisson case was obtained using $E(x) = \lambda$ while that for the Geometric case is $E(x) = {}^{1}/_{P}$.

*B. **Results and Discussion.***

The optimal solution for the individual replenishment policy using the multi-item EOQ model with probabilistic demand using EXCEL using EXCEL is presented in table 2. The lead time for the problem is 7 days. The optimal order quantity and reorder point values are also given in table 3 together with the total purchasing cost, total ordering cost, total holding cost and total shortage cost.
The total inventory cost for the individual replenishment policy is ₦16,196,193.00.

For the joint replenishment policy, using the joint ordering as $S^* = 0.6\sum_{i=1}^{n} S_i$, $T = 0.16$ **,** the summary of the results for Winjen Services using the modified algorithm in [2] is shown in table 4. From the results, the optimal ordering time is 0.16 years while the optimal quantity for the products are 259, 283, 264,238 and 199 respectively. These are less than the optimal order quantity for the individual replenishment policy except for Mirinda. This of course has an impact on the holding and shortage costs.

Table 2. Results on fitting distribution on previous demand for the drinks and their current expected demand.

		7up drink		**Mirinda**		**Pepsi**		**Soda water**		**Sprite**	
	Distribution	**Anderson Darling Test**		**Anderson Darling Test**		**Anderson Darling Test**		**Anderson Darling Test**		**Anderson Darling Test**	
		Statistic	Rank	Statistic	Rank	Statistic	Rank	Statistic	Rank	Statistic	Rank
1	D. Uniform	3.7501	3	3.8117	3	3.722	3	3.9742	3	3.907	3
2	Geometric	4.0115	4	0.3977	1	3.7905	4	0.50926	1	0.34333	1
3	Logarithmic	8.4745	5	7.2485	4	7.8262	5	7.8609	4	7.8609	5
4	Neg. Binomial	0.3254	2	2.1668	2	1.6713	2	8.6511	5	3.3525	2
5	Poisson	0.23216	1	29.321	5	0.269	1	3.1612	2	5.4555	4
6	Binomial	No fit		No fit		No fit		No fit		No fit	
7	Hyper geometric	No fit		No fit		No fit		No fit		No fit	
Distribution of demand for the drinks		Poisson		Geometric		Poisson		Geometric		Geometric	
Parameter		λ = 134.92		p = 0.0068		λ = 133.6		p = 0.0096		P = 0.0073	
$E(x)$		134.9≈ 135		146.9≈ 147		133.6≈ 134		103.9≈ 104		136.4≈ 136	

Table 3. Summary of results for Individual Replenishment Cost.

	Products					
	7UP	MIRINDA	PEPSI	SPRITE	SODA	TOTAL
Q	273	254	262	275	237	
R	76	79	74	56	67	
Total Demand	1619	1762	1649	1488	1245	
Total Purchase Cost	₦1919350	₦2246900	₦1978350	₦1711150	₦1431750	₦9,287,500
Total Ordering Cost	₦1193522.50	₦1168625.09	₦1087927.81	₦642,030.26	₦924,191.46	₦5,016,297
Total Holding Cost	₦361828.05	₦443,769.6	₦392,743.61	₦270,963	₦243,977.63	₦1,713,282
Total Shortage Cost	36855.13	41275.20	44057.25	20825.45	36101.27	₦179,114.3
Total	₦3,511,556	₦3,900,569.89	₦3,503,078.67	₦2,644,968.71	₦2,636,020.36	
TAC(Q,R) = **₦16,196,193.00**						

Table 4. Summary of results for the joint replenishment policy.

JOINT REPLENISHMENT POLICY						
PRODUCTS	7UP	MIRINDA	PEPSI	SPRITE	SODA	TOTAL
T(year)	0.16					
Joint ordering cost (S^*)	₦187,284					₦187,284
Q	259	282	264	238	199	
Total purchase cost (P_iD_i)	1919352.64	2246902.4	1978354.77	1431750	1711155	₦9,287,515.17
P_ih_i	238.3584	318.8	263.9406	195.5	183.9952	
Total holding cost $\frac{TD_i}{2} + R_i - E(x_i)$	29861.54	41431.25	32971.46	15170.8	20430.83	₦139,865.88
$\frac{\pi_i}{T}$	37187.5	37187.5	43750	28750	35937.5	
$E(x_i) - R_i$	10	11	7	22	8	
Total shortage cost	371875	409062.5	306250	632500	287500	₦2,007,187.5
TAC	₦12,605,093.55					

The holding cost of the IRP which is ₦1,713,282 is higher than that of the JRP which is ₦139,865.88 since we have more products at hand in the IRP than JRP. Also the shortage cost of the IRP which is ₦179,114.3 is lower than that of the JRP which is ₦2,007,187.5 since we have more shortage in the JRP than IRP. Overall though, between the two policies, we can see that the total inventory cost of the IRP, which is ₦16,196,193.00, is higher than the total inventory cost of the JRP which is ₦12,605,093.55.

Hence we can say the Joint Replenishment Policy is better than the individual Replenishment Policy for the distributions of the demand obtained for the different drinks. This also corresponds to the results obtained in [2] with a Gamma distributed lead time demand.

IV. Conclusion

In this paper, a multi-item probabilistic inventory model with discrete and probabilistic demand in a fixed reorder and zero lead-time is presented. The model was derived from the EOQ individual and joint replenishment model with Gamma lead time demand in [2] but lead time demand was not assumed to be Gamma distributed but has an unknown distribution. The model was for a multi-item ordering/inventory problem The objective was to determine the optimal ordering quantity that minimizes the total cost for each product i.e. Individual Replenishment Policy (IRP) and also the optimal ordering time for all the products i.e. Joint Replenishment Policy (JRP). The models were illustrated with data collected from an ordering/inventory firm Winjec Services in Enugu state. The total costs considered included the purchasing cost, ordering cost, handling and shortage cost. The data is analyzed using EXCEL. From the solution obtained the holding cost of the IRP which is ₦1,713,282 is higher than that of the JRP which is ₦139,865.88 since we have more products at hand in the IRP than JRP. Also the shortage cost of the IRP which is ₦179,114.3 is lower than that of the JRP which is ₦2,007,187.5 since we have more shortage in the JRP than IRP. Overall though, between the two policies, we can see that the total inventory cost of the IRP which is ₦16,196,193.00 is higher than the total inventory cost of the JRP of ₦12,605,093.55. Hence we can conclude that the Joint Replenishment Policy is better than the individual Replenishment Policy with respect with the demand distribution of the data used.

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