Burr X-Kumaraswamy Distribution: Properties and Application

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***Abstract* — This research studied and explored the Burr X-Kumaraswamy distribution using the family of Burr type X However, we developed a new continuous distribution called Burr X-Kumaraswamy distribution which extends the Kumaraswamy distribution. We obtained the density and distribution function of the proposed distribution. Some structural properties of the proposed distribution were derived such as moment, moment generating function, quantile function and order statistics. We estimate parameters by using Maximum Likelihood methods. Finally, the obtained results are validated using a real data set and is shown that the new distribution provides a better fit than some other related known distributions. This new distribution will serve as an alternative model to other models available in the literature for modeling positive real data sets.**

Keywords - *Burr X Kumaraswamy; Burr Type X Faimly; moment generating function; quantile function; order statistics; maximum likelihood estimation.*

I. Introduction

The building of new distributions allows more flexibility to model real data that posses a high degree of skewness and kurtosis. In doing this, new techniques are developed by adding more parameter(s) to the existing and traditional distributions for developing more classes of flexible distributions. The new distribution proposes in this work is Burr type X- Kumaraswamy. Burr type X distribution is used as the generator [6], while, Kumaraswamy is the baseline distribution. However, this new family of distribution (Burr type X) has been used to extend the Exponential distribution [15], Weibull and Lomax distributions [6].

However, Burr X model distribution was proposed by [3] and revisited by [16]-[17], who contribute a lot to the family of continuous distributions and played a vital role in medical, survival and reliability analysis. Burr type X model distribution is very effective and versatile in modeling reliability strength and lifetime data sets. In fact, many authors have studied Burr type X distribution behavior, such as [1], [2], [8], [9], [12] and [13]. Perhaps, some well known distributions that are family of exponential are typically related to Burr type X distribution.

Thus, Kumaraswamy's double bounded distribution is a family of continuous probability defined on the interval [0,1]. It is similar to the Beta distribution, but much simpler to use especially in simulation studies due to the simple closed form of both its probability density function and cumulative distribution function. This distribution was originally proposed by Poondi Kumaraswamy for variables that are lower and upper bounded [10]. However, kumaraswamy distribution has been used a base line distribution in [7]. However, the cdf and pdf of the Burr X family of distribution proposed by [6], is given by;

$$F(x)=\left[1-\exp\left\{-\left(\frac{G(x)}{1-G(x)}\right)^2\right\}\right]^{\theta} \quad (1)$$

$$f(x)=\frac{2\theta g(x)G(x)}{\left(1-G(x)\right)^2}\exp\left\{-\left(\frac{G(x)}{1-G(x)}\right)^2\right\}\left[1-\exp\left\{-\left(\frac{G(x)}{1-G(x)}\right)^2\right\}\right]^{\theta-1} \quad (2)$$

respectively for $x>0, \theta>0$. Where; θ is the shape parameter whose role is to vary the tail weight. $G(x)$ and $g(x)$ are the cdf and pdf of the baseline distribution respectively. The baseline distribution used in this research work is Kumaraswamy distribution. The cdf and pdf of the baseline distribution are given in (3) and (4) respectively.

$$G(x) = 1-\left(1-x^c\right)^d \quad (3)$$

$$g(x) = cdx^{c-1}\left(1-x^c\right)^{d-1} \quad (4).$$

II. The New Burr X-Kumaraswamy Distribution

The cdf and pdf of the proposed distribution with three parameters are defined respectively by:

$$F(x;c,d,\theta) = \left[1-\exp\left\{-\left(\frac{1-\left(1-x^c\right)^d}{\left(1-x^c\right)^d}\right)^2\right\}\right]^\theta \quad (5)$$

$$f(x;c,d,\theta) = \frac{2\theta cdx^{c-1}\left(1-x^c\right)^{d-1} 1-\left(1-x^c\right)^d}{\left(\left(1-x^c\right)^d\right)^2} \times$$

$$\exp\left\{-\left(\frac{1-\left(1-x^c\right)^d}{\left(1-x^c\right)^d}\right)^2\right\}\left[1-\exp\left\{-\left(\frac{1-\left(1-x^c\right)^d}{\left(1-x^c\right)^d}\right)^2\right\}\right]^{\theta-1} \quad (6)$$

For $x > 0, \theta, c, d > 0$. Where; θ, c and d are all shape parameters.

However, the corresponding, survival rate S(x), hazard or failure rate h(x) and cumulative hazard functions H(x) are respectively given by:

$$S(x;\theta,c,d) = \left[1-\left[1-\exp\left\{-\left(\frac{1-\left(1-x^c\right)^d}{\left(1-x^c\right)^d}\right)^2\right\}\right]^\theta\right] \quad (7)$$

$$h(x;c,d,\theta) = 2\theta cdx^{c-1}\left(1-x^c\right)^{d-1}\left(1-\left(1-x^c\right)^d\right)\times$$

$$\exp\left\{-\left(\frac{1-\left(1-x^c\right)^d}{\left(1-x^c\right)^d}\right)^2\right\}\left[1-\exp\left\{-\left(\frac{1-\left(1-x^c\right)^d}{\left(1-x^c\right)^d}\right)^2\right\}\right]^{\theta-1}\times$$

$$\frac{1}{\left(\left(1-x^c\right)^d\right)^2\left[1-\left[1-\exp\left\{-\left(\frac{1-\left(1-x^c\right)^d}{\left(1-x^c\right)^d}\right)^2\right\}\right]^\theta\right]} \quad (8)$$

$$H(x;\theta,c,d) = -\left[\log\left[1-\exp\left\{-\left(\frac{1-\left(1-x^c\right)^d}{\left(1-x^c\right)^d}\right)^2\right\}\right]\right]^\theta \quad (9)$$

Shapes of the Burr X – Kumaraswamy (BX-K) Distribution

Figure (1) and Figure (2), describes the shapes of the probability density function and cumulative distribution function for the initial given values of the parameters. These functions represent a different kind of forms depending on choosing the values of the parameters. We noticed that the additional shape parameters allow for a high level of flexibility.

On the other hand, Figure (3) and Figure (4) below, shows the survival function and hazard function of BX-K model increasing and decreasing shapes.

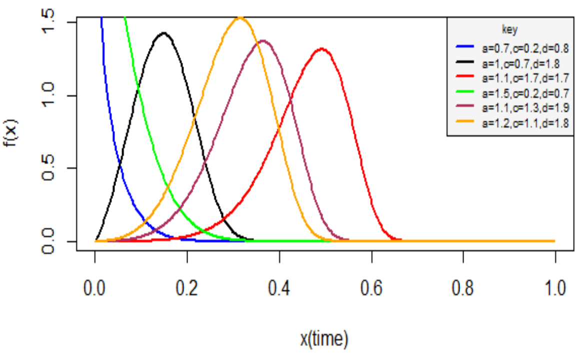

Figure 1: Plot of the pdf of Burr X Kumaraswamy

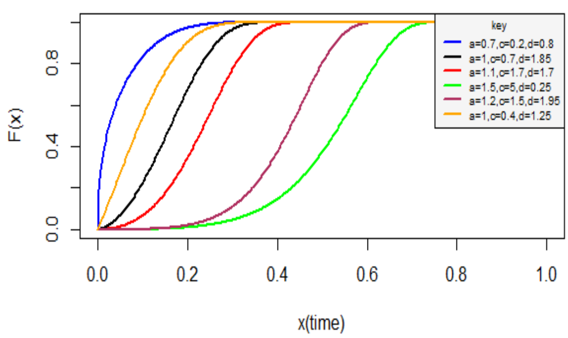

Figure 2 Plot of the cdf of Burr X Kumaraswamy

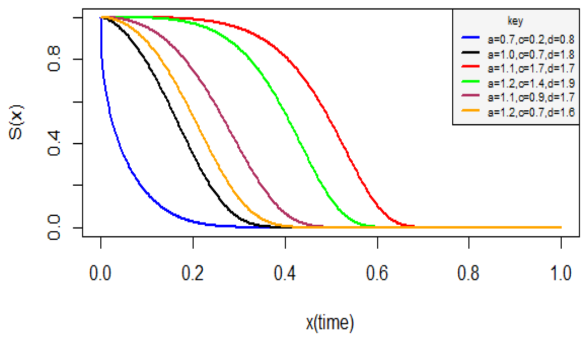

Figure 3: Plot of the Survival function of Burr X Kumaraswamy

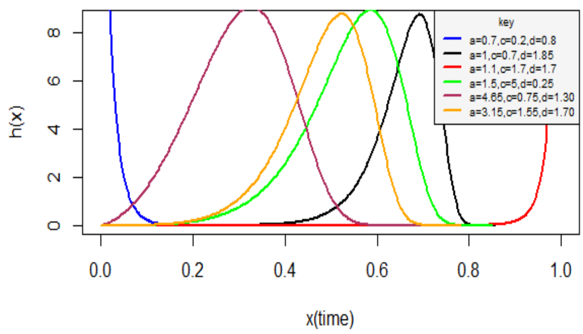

Figure 4: Plot of the Hazard function of Burr X Distribution

III. Some Properties Of The Burr X-Kumaraswamy Distribution

3.1 Moment & Moment Generating Functions

To get the MGF of the new distribution, we present the linear representation; this is done by expanding the equation of the probability density function (pdf) of the generator from equation (2). According to Binomial expansion in [4], if $[W]<1$ and $[\omega]>0$ is a real non-integer, we have the series representation:

$$(1-W)^{\omega-1}=\sum_{p=0}^{\infty}(-1)^{p}\frac{\Gamma(\omega)}{p!\Gamma(\omega-p)}W^{p} \qquad (10)$$

Thus equation (4) becomes

$$f(x)=\frac{2\theta\sum_{p=0}^{\infty}(-1)^{p}\Gamma\theta g(x)G(x)}{p!\Gamma(\theta-p)(1-G(x))^{3}}\times \exp\left\{-(p+1)\left(\frac{G(x)}{1-G(x)}\right)^{2}\right\} \qquad (11)$$

$$f(x)=\frac{2\theta\sum_{p,q=0}^{\infty}(-1)^{p+q}(p+1)^{q}\Gamma\theta g(x)G(x)^{1+2q}}{p!q!\Gamma(\theta-p)(1-G(x))^{3+2q}} \qquad (12)$$

$$=2\theta\sum_{p,q,r=0}^{\infty}(-1)^{p+q}\frac{\Gamma(\theta)(p+1)^{q}\Gamma(3+2p+r)}{q!p!\Gamma(\theta-p)\Gamma(3+2p)}\times \frac{2+r+2q}{2+r+2q}g(x)G(x)^{1+r+2q} \qquad (13)$$

$$=\sum_{p.q,r=0}^{\infty}\Phi\Psi_{2+r+2q} \qquad (14)$$

Where: $\Phi=\dfrac{2\theta(-1)^{p+q}(p+1)^{q}\Gamma(\theta)\Gamma(3+2p+r)}{q!p!\Gamma(\theta-p)\Gamma(3+2p)(2+r+2q)}$

And $\Psi_{2+r+2q}=(2+r+2q)g(x)G(x)^{1+r+2q+1-1}$

Thus, the linear representation is presented in equation (14) above. The r^{th} moment about the origin for the new distribution is defined as:

$$E(x^{r})=\int_{-\infty}^{\infty}x^{r}f(x)dx \qquad (15)$$

By substituting the probability density function (pdf) and cumulative density function (cdf) of Kumaraswamy in $f(x)$ as defined in (15) above, we have:

$$=\sum_{p,q,r=0}^{\infty}\Phi(2+r+2q)\int_{0}^{1}x^{r}\left[cdx^{c-1}(1-x^{c})^{d-1}\right]\times \left[1-(1-x^{c})^{d}\right]^{1+r+2q}dx \qquad (16)$$

$$=\sum_{p,q,r,s=0}^{\infty}\Phi(2+r+2q)cd(-1)^{s}\binom{1+r+2q}{s}\times \int_{0}^{1}x^{r+c-1}(1-x^{c})^{d(s+1)-1}dx \qquad (17)$$

$$=\varphi\int_{0}^{1} y^{\frac{r}{c}+1-1}(1-y)^{d(s+1)-1}\,dx \qquad (18)$$

Therefore:

$$E\left(x^{r}\right)=\varphi\beta\left(\frac{r}{c}+1,d(s+1)\right) \qquad (19)$$

Where:

$$\varphi=\frac{\sum_{p,q,r,s=0}^{\infty}\Phi(2+r+2q)cd(-1)^{s}\binom{1+r+2q}{s}}{c}$$

3.2 Quantile function

The quantile function (qf) corresponding to equation (5) is obtained by inverting it as

$$F=\left[1-\exp\left\{-\left(\frac{1-\left(1-x^{c}\right)^{d}}{\left(1-x^{c}\right)^{d}}\right)^{2}\right\}\right]^{\theta}=U \qquad (20)$$

Simulating of Burr X-Kumaraswamy Distribution random variable is straight forward, if U has a uniform variate on the interval (0, 1). By means of the inverse transformation method, the random variable X is given as:

$$x=\quad 1-\left(\frac{1}{\left[\left[1-\left[\log\left(1-U^{\frac{1}{\theta}}\right)\right]^{\frac{1}{2}}\right]^{\frac{1}{d}}\right]^{\frac{1}{c}}}\right) \qquad (21)$$

Skewness and Kurtosis

The Bowley's skewness [11] is based on the quartiles and is defined as

$$S_{K}=\frac{Q\left(3/4\right)-2Q\left(1/2\right)+Q\left(1/4\right)}{Q\left(3/4\right)-Q\left(1/4\right)} \qquad (22)$$

And the Moor's Kurtosis [14] is based on percentiles and is given as:

$$M_{K}=\frac{Q\left(7/8\right)-Q\left(5/8\right)-Q\left(3/8\right)+Q\left(1/8\right)}{Q\left(6/8\right)-Q\left(2/8\right)} \qquad (23)$$

3.3. Order Statistics

$$f_{i,n}(x)=\frac{n!}{(i-1)!(n-i)!}f(x)F(x)^{i-1}\left[1-F(x)\right]^{n-1} \qquad (24)$$

By substituting *f(x)* and *F(x)* in (24) from (5) and (6) with respective expansion we have:

$$=\frac{n!}{(i-1)!(n-i)!}\frac{2\theta cdx^{c-1}\left(1-\left(1-x^{c}\right)^{d}\right)}{\left(\left(1-x^{c}\right)^{d}\right)^{3}}\times$$

$$\sum_{p,q,r,s,k=0}^{\infty}(-1)^{p+q+r+s+t}\frac{(n-i)!\Gamma(\theta)}{p!(n-i-p)!q!}\times$$

$$\frac{\theta\Gamma(i)\theta p}{\Gamma(\theta-p)r!\theta\Gamma(i-r)s!(\theta p-s)!k!}\times$$

$$\left[-\left(\frac{1-\left(1-x^{c}\right)^{d}}{\left(1-x^{c}\right)^{d}}\right)^{2}\right]^{(1+q+r+s)k} \qquad (25)$$

IV. Estimation And Inference

The most widely used method for the estimation of parameters of distribution is the maximum likelihood estimation (MLE) and the moment method. We employ MLE to estimate the unknown parameter of BX-K distribution. The BX-K model with three parameters, θ, c, d. Given the likelihood function as:

$$L(\theta,c,d)=\prod_{i=1}^{n}\left(2\theta cdx^{c-1}(1-x)^{d-1}\left(1-\left(1-x^{c}\right)^{d}\right)\right)\times$$

$$\left(\left(\left(1-x^{c}\right)^{d}\right)^{-3}\times e^{-y}\times\left(1-e^{-y}\right)^{\theta-1}\right) \qquad (26)$$

where:

$$y=\left(\frac{1-\left(1-x^{c}\right)^{d}}{\left(1-x^{c}\right)^{d}}\right)^{2} \quad \text{For } \theta,c,d\succ 0;$$

Let $x=\left(x_1;x_2;\ldots;x_n\right)^x$ be a random sample of size "n" from BX-K with parameters: $\left(\theta;c;d\right)^X$, for i=1,2,...,n. Then the log-likelihood function for is given by:

$$=n\ln\left(2\theta cd\right)+\left(c-1\right)\sum_{i=1}^{n}\ln\left(x\right)+$$
$$\left(d-1\right)\sum_{i=1}^{n}\ln\left(1-x^c\right)+\sum_{i=1}^{n}\ln\left(1-\left(1-x^c\right)^d\right)$$
$$-3\sum_{i=1}^{n}\ln\left(\left(1-x^c\right)^d\right)-\sum_{i=1}^{n}y+$$
$$\left(\theta-1\right)\sum_{i=1}^{n}\ln\left(1-e^{-y}\right) \qquad (27)$$

Differentiating Equation (43) with respect to parameters, θ, c *and* d, setting the resulting non-linear system of equations to zero and solving them give the maximum likelihood estimates of parameters θ, c *and* d respectively. It is much easier to solve these equations using algorithms in statistical software like R and so on when data sets are available.

V. Applications

In this section, we illustrate the usefulness of the proposed Burr X-Kumaraswamy distribution. We fit this distribution to a real life data set and compare the results with other distributions such as Burr X distribution and Kumaraswamy distribution.

Data set

The data set is based on the flood data with 20 observations studied by [5] and it's given below:

0.265, 0.269, 0.297, 0.315, 0.3235, 0.338, 0.379, 0.379, 0.392, 0.402, 0.412, 0.416, 0.418, 0.423, 0.449, 0.484, 0.494, 0.613, 0.654, 0.74

Table 1 shows the Maximum Likelihood Estimate and Log-like hood (ℓ) estimate of the model parameters

In order to compare the proposed model with order models such as Burr X distribution and Kumaraswamy distribution, we consider the criteria; Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (AICC), and Bayesian Information Criterion (BIC) for the data set. In general, the smaller the value of these criteria, the better the fit of the data.

Table 1: The MLE and Log-like hood (ℓ) estimate of the model parameters.

Distribution	Parameters			ℓ
	a	b	θ	
BX	-	-	0.529	15.7061
Kum.	0.001	0.072	-	52.0959
BX – Kum.	0.010	2.833	4.700	119.295

Table 2: The AIC, AICC and BIC for flood data

Distribution	AIC	AICC	BIC
BX	-29.412	-27.912	-27.5092
Kum.	-100.191	-98.691	-131.701
BX – Kum.	-232.590	-231.090	-370.2912

Note: The short form of the distributions used are Burr X is ''BX'', Kumaraswamy is ''Kum.'' and Burr X Kumaraswamy is ''BX - Kum.

From Table 1, it shows that the proposed Burr X-Kumaraswamy model has a maximum value of log likelihood. Table 2 shows that the proposed model has a minimum values of statistics compared to other models. As the results indicate, the proposed model performed better than other models in terms of its flexibility.

VI. Conclusion

In this paper, we proposed a new lifetime model called Burr X-Kumaraswamy distribution (BX-KD) which extends the Kumaraswamy distribution. We defined the probability density function, cumulative distribution function, survival and hazard functions of the proposed distribution. We provided some of its statistical properties including moments, moment generating function, quantile function, skewness, kurtosis and order statistics of Burr X Kumaraswamy Distribution.

The maximum likelihood method was studied. We fit the Burr X Kumaraswamy model to a real data set to demonstrate the flexibility and potentiality of the distribution. Our finding shows that the proposed model performed better than the other two compared models (i.e. Burr X Model and Kumaraswamy Model). We hope that this distribution would attract wider applications in reliability in engineering, or medical or finance or actuarial or hydrology or any other related areas to model lifetime data.

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