Efficient Distance Function for Calibrating Estimators in Stratified Random Sampling

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***Abstract* — Calibration weights are derived by minimizing distance function which satisfied a set of constraints in line with the auxiliary information. In this study, existing different distance functions are put into consideration and new calibration weights and new calibration estimators of population mean are obtained in stratified random sampling. A simulation study is carried out to ascertain the most efficient among the distance functions. The results revealed that distance function Z_4 performed better than other distance functions considered.**

Keywords: *Distance Function, Calibration Weight, Stratified Sampling, Auxiliary Information.*

I. Introduction

Calibration estimation is a general technique of modifying the original weights with minimization of a given distance measure based on a set of calibration constraints under auxiliary information. In literature, researchers tried to increase estimates of the population parameter with constructing new calibration weights in stratified sampling. Calibration estimation adjust the original design weights to incorporate the known population totals of auxiliary variables. The calibration weights are chosen to minimize a given distance measure and these weight satisfy the constraints related auxiliary variable information. Calibration estimator uses calibrated weights that are aimed to minimize a given distance measure to the original design weight while satisfying a set of constraints related to the auxiliary information.

The technique of estimation by calibration in survey sampling was introduced by Deville and Sarndal in 1992. The idea is to use auxiliary information to obtain a better estimate of a population statistic. They state that the calibrated weights may give perfect estimates when the sample sum of the weighted auxiliary variable is equal the known population total for that auxiliary variable.

Following Deville and Sarndal (1992), many researchers have studied calibration estimation with using different calibration constraints in survey sampling design. Singh *et al* (1998) is the first researcher that extended calibration approach to a stratified sampling design. Singh (2003), Tracy *et al* (2003), Kim *et al* (2007), Rao *et al* (2012), Clement (2015), Koyuncu and Kadilar (2016), Lata *et al* (2017), Ozgul (2018), Garg and Pachori, (2019), applied calibration estimation to ratio-type estimators in stratified sampling using different calibration constraints based on auxiliary information.

II. Notations and Review of Existing Estimators

Consider a finite population ψ of N elements, $\psi=\{\psi_1,\psi_2,\psi_3,...,\psi_N\}$ consists of L strata with N_h units in the hth stratum from which a simple random of size n_h is taken from the population using SRSWOR. Total Population size $N=\sum_{h=1}^{L}N_h$, sample size $n=\sum_{h=1}^{L}n_h$ where $y_{hi}, i=1,2,...,N_{hi}$ and $x_{hi}, i=1,2,...,N_{hi}$ of study variable y and x auxiliary variable. Let $W_h=N_h/N$ be the strata weights, $\bar{y}_h=n^{-1}\sum_{i=1}^{n_h}y_{hi}$ and $\bar{Y}_h=N^{-1}\sum_{i=1}^{n_h}y_{hi}$ are the sample and population means respectively for the study variables.

According to Cochran (1977), the traditional estimator of population mean in stratified sampling given as:

$$\bar{y}_{st} = \sum_{h=1}^{L} W_h \bar{y}_h \tag{1.1}$$

$$V\left(\bar{y}_{st}\right) = \sum_{h=1}^{L} W_h^2 \left(\frac{1-f_h}{n_h}\right) s_{hy}^2 \tag{1.2}$$

where $s_{hy}^2 = \left(n_h - 1\right)^{-1} \sum_{h=1}^{n_h} \left(y_{hi} - \bar{y}_h\right)^2,\ f_h = n_h / N_h$

Alam *et al*. (2019) developed a calibration estimator for estimating population mean under stratified random sampling using a distance function as follows:

$$\bar{y}_{st}^{A} = \sum_{h=1}^{L} \Omega_h^A \bar{y}_h \tag{1.3}$$

where Ω_h^A are the calibration weights which are chosen with the distance function

$$Z = \frac{1}{2} \sum_{h=1}^{L} \frac{\left(\Omega_h^A - W_h\right)^2}{W_h Q_h} + \sum_{h=1}^{L} \sum_{h \neq h^l = 1}^{L} \left(\Omega_h^A - W_h\right)\left(W_{h^l}^{(1)} - W_{h^l}\right) \tag{1.4}$$

where $W_{h^l}^{(1)}$ is the $h^{l\,th}$ stratum weight which minimum subject to the calibration constraints:

$$\sum_{h=1}^{L} \Omega_h^A = \sum_{h=1}^{L} W_h \tag{1.5}$$

$$\sum_{h=1}^{L} \Omega_h^A \bar{x}_h = \bar{X} \tag{1.6}$$

Minimization of (1.4) subject to the calibration constraint given in (1.5) and (1.6), the calibration weights are given by

$$\Omega_h^A = W_h + \lambda_1 \left(\frac{W_h Q_h}{1 - W_h Q_h}\right) + \lambda_2 \left(\frac{W_h Q_h \bar{x}_h}{1 - W_h Q_h}\right) \tag{1.7}$$

And the calibrated estimator is

$$\bar{y}_{st}^{A} = \sum_{h=1}^{L} W_h \bar{y}_h + \hat{\beta}_a \left(\bar{X} - \sum_{h=1}^{L} W_h \bar{x}_h\right) \tag{1.8}$$

$$\hat{\beta}_a = \frac{\sum_{h=1}^{L}\left(\frac{W_h Q_h \bar{x}_h \bar{y}_h}{1-W_h Q_h}\right)\sum_{h=1}^{L}\left(\frac{W_h Q_h}{1-W_h Q_h}\right) - \sum_{h=1}^{L}\left(\frac{W_h Q_h \bar{x}_h}{1-W_h Q_h}\right)\sum_{h=1}^{L}\left(\frac{W_h Q_h \bar{y}_h}{1-W_h Q_h}\right)}{\sum_{h=1}^{L}\left(\frac{W_h Q_h}{1-W_h Q_h}\right)\sum_{h=1}^{L}\left(\frac{W_h Q_h \bar{x}_h^2}{1-W_h Q_h}\right) - \sum_{h=1}^{L}\left(\frac{W_h Q_h \bar{x}_h}{1-W_h Q_h}\right)^2} \tag{1.9}$$

2.1 Distance Function

The following distance functions are considered

$$Z_1 = \sum_{h=1}^{L} \frac{\left(\Omega_h^S - W_h\right)^2}{Q_h W_h} \tag{1.10}$$

$$Z_2 = \sum_{h=1}^{L} S_{hx}^2 \left(Q_h\right)^{-1} \left(\Omega_h^{LA} - W_h\right)^2 \tag{1.11}$$

$$Z_3 = \sum_{h=1}^{L} \frac{W_h}{Q_h} \left(\frac{\Omega_h^A}{W_h} - 1\right)^2 \tag{1.12}$$

$$Z_4 = \frac{1}{2} \sum_{h=1}^{L} \frac{\left(\Omega_h^A - W_h\right)^2}{W_h Q_h} + \sum_{h=1}^{L} \sum_{h \neq h^l = 1}^{L} \left(\Omega_h^A - W_h\right)\left(W_{h^l}^{(1)} - W_{h^l}\right) \tag{1.13}$$

$$Z_5 = \sum_{h=1}^{L} \frac{1}{Q_h} \left(\frac{\Omega_h^A}{W_h} - 1\right)^2 \tag{1.14}$$

2.1.1 First Distance Function $\left(Z_1\right)$

Applying Lagrange multipliers method to minimize (1.10), presence of the constraints given in (1.5) and (1.6), we have the Lagrange function as

$$L_1 = \sum_{h=1}^{L} \frac{\left(\Omega_h - W_h\right)^2}{W_h Q_h} - 2\lambda_1 \left(\sum_{h=1}^{L} \Omega_h^A - \sum_{h=1}^{L} W_h\right) - 2\lambda_2 \left(\sum_{h=1}^{L} \Omega_h^A \bar{x}_h - \bar{X}\right) \tag{1.15}$$

where λ_1 and λ_2 are the Lagrange multipliers. To find Ω_h^A, (1.15) is partially differentiate with respect to Ω_h^A and equating to zero as:

$$\frac{\partial L_1}{\partial \Omega_h^A} = 2\frac{\left(\Omega_h^A - W_h\right)}{W_h Q_h} - 2\lambda_1 - 2\lambda_2 \bar{x}_h = 0 \tag{1.16}$$

From (1.16), we have

$$\Omega_h^A = W_h + \lambda_1 W_h Q_h + \lambda_2 W_h Q_h \bar{x}_h = 0 \tag{1.17}$$

Using (1.17), (1.5) and (1.6) then λ_1 and λ_2 are

$$\lambda_1 = -\frac{\left(\bar{X}_h - \sum_{h=1}^{L} W_h \bar{x}_h\right)\left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h\right)}{\left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h^2\right)\left(\sum_{h=1}^{L} W_h Q_h\right) - \left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h\right)^2} \tag{1.18}$$

$$\lambda_2 = \frac{\left(\bar{X}_h - \sum_{h=1}^{L} W_h\right)\left(\sum_{h=1}^{L} W_h Q_h\right)}{\left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h^2\right)\left(\sum_{h=1}^{L} W_h Q_h\right) - \left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h\right)^2} \tag{1.19}$$

Substituting (1.18) and (1.19) in (1.17), the new calibration weights can be written as

$$\Omega_h^A = W_h - \frac{\left(\bar{X}_h - \sum_{h=1}^{L} W_h \bar{x}_h\right)\left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h\right)}{\left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h^2\right)\left(\sum_{h=1}^{L} W_h Q_h\right) - \left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h\right)^2}\left(W_h Q_h\right) + \frac{\left(\bar{X}_h - \sum_{h=1}^{L} W_h \bar{x}_h\right)\left(\sum_{h=1}^{L} W_h Q_h\right)}{\left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h^2\right)\left(\sum_{h=1}^{L} W_h Q_h\right) - \left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h\right)^2}\left(W_h Q_h \bar{x}_h\right) \tag{1.20}$$

Substituting (1.20) in (1.3), the new calibration estimator is

$$\bar{y}_{st}^* = \sum_{h=1}^{L} W_h \bar{y}_h + \hat{\beta}\left(\bar{X} - \sum_{h=1}^{L} W_h \bar{x}_h\right) \tag{1.21}$$

where

$$\hat{\beta} = \frac{\left(\sum_{h=1}^{L} W_h Q_h \bar{y}_h\right)\left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h\right) - \left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h \bar{y}_h\right)\left(\sum_{h=1}^{L} W_h Q_h\right)}{\left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h^2\right)\left(\sum_{h=1}^{L} W_h Q_h\right) - \left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h\right)^2}$$

2.1.2 Second Distance Function $\left(Z_2\right)$

Applying Lagrange multipliers method to minimize (1.11), presence of the constraints given in (1.5) and (1.6), we have the Lagrange function as

$$L_2 = \sum_{h=1}^{L} S_{hx}^2 \left(Q_h\right)^{-1}\left(\Omega_h^M - W_h\right)^2 - 2\lambda_3\left(\sum_{h=1}^{L} \Omega_h^M - \sum_{h=1}^{L} W_h\right) - 2\lambda_4\left(\sum_{h=1}^{L} \Omega_h^M \bar{x}_h - \bar{X}\right)$$

(1.22)

where λ_3 and λ_4 are the Lagrange multipliers. To find Ω_h^M, (1.22) is partially differentiate with respect Ω_h^M to and equating to zero as:

$$\frac{\partial L_1}{\partial \Omega_h^A} = 2S_{hx}^2\left(\Omega_h^M - W_h\right)\left(Q_h\right)^{-1} - 2\lambda_1 - 2\lambda_2 \bar{x}_h = 0 \tag{1.23}$$

From (1.23), we get

$$\Omega_h^M = W_h + \lambda_3\left(S_{hx}^2\right)^{-1} Q_h + \lambda_4\left(S_{hx}^2\right)^{-1} Q_h \bar{x}_h = 0 \tag{1.24}$$

Using (1.24), (1.15) and (1.16) then λ_1 and λ_2 are

$$\lambda_3 = -\frac{\left(\bar{X}_h - \sum_{h=1}^{L} W_h \bar{x}_h\right)\left(\sum_{h=1}^{L}\left(S_{hx}^2\right)^{-1} Q_h \bar{x}_h\right)}{\left(\sum_{h=1}^{L}\left(S_{hx}^2\right)^{-1} Q_h\right)\left(\sum_{h=1}^{L}\left(S_{hx}^2\right)^{-1} Q_h \bar{x}_h^2\right) - \left(\sum_{h=1}^{L}\left(S_{hx}^2\right)^{-1} Q_h \bar{x}_h\right)^2} \tag{1.25}$$

$$\lambda_4 = \frac{\left(\bar{X}_h - \sum_{h=1}^{L} W_h \bar{x}_h\right)\left(\sum_{h=1}^{L}\left(S_{hx}^2\right)^{-1} Q_h\right)}{\left(\sum_{h=1}^{L}\left(S_{hx}^2\right)^{-1} Q_h\right)\left(\sum_{h=1}^{L}\left(S_{hx}^2\right)^{-1} Q_h \bar{x}_h^2\right) - \left(\sum_{h=1}^{L}\left(S_{hx}^2\right)^{-1} Q_h \bar{x}_h\right)^2} \tag{1.26}$$

Substituting (1.25) and (1.26) in (1.24), the new calibration weights can be written as

$$\Omega_h^M = W_h - \frac{\left(\bar{X}_h - \sum_{h=1}^{L} W_h\right)\left(\sum_{h=1}^{L}\left(S_{hx}^2\right)^{-1} Q_h \bar{x}_h\right)}{\left(\sum_{h=1}^{L}\left(S_{hx}^2\right)^{-1} Q_h\right)\left(\sum_{h=1}^{L}\left(S_{hx}^2\right)^{-1} Q_h \bar{x}_h^2\right) - \left(\sum_{h=1}^{L}\left(S_{hx}^2\right)^{-1} Q_h \bar{x}_h\right)^2}\left(\left(S_{hx}^2\right)^{-1} Q_h\right) + \frac{\left(\bar{X}_h - \sum_{h=1}^{L} W_h\right)\left(\sum_{h=1}^{L}\left(S_{hx}^2\right)^{-1} Q_h\right)}{\left(\sum_{h=1}^{L}\left(S_{hx}^2\right)^{-1} Q_h\right)\left(\sum_{h=1}^{L}\left(S_{hx}^2\right)^{-1} Q_h \bar{x}_h^2\right) - \left(\sum_{h=1}^{L}\left(S_{hx}^2\right)^{-1} Q_h \bar{x}_h\right)^2}\left(\left(S_{hx}^2\right)^{-1} Q_h \bar{x}_h\right) \tag{1.27}$$

Substituting (1.27) in (1.3), the new calibration estimator is

$$\bar{y}_{st}^M = \sum_{h=1}^{L} W_h \bar{y}_h + \hat{\beta}_m\left(\bar{X} - \sum_{h=1}^{L} W_h \bar{x}_h\right)$$

(1.28)

where

$$\hat{\beta}_m = \frac{\left(\sum_{h=1}^{L}\left(S_{hx}^2\right)^{-1} Q_h \bar{y}_h\right)\left(\sum_{h=1}^{L}\left(S_{hx}^2\right)^{-1} Q_h \bar{x}_h\right) - \left(\sum_{h=1}^{L}\left(S_{hx}^2\right)^{-1} Q_h \bar{x}_h \bar{y}_h\right)\left(\sum_{h=1}^{L}\left(S_{hx}^2\right)^{-1} Q_h\right)}{\left(\sum_{h=1}^{L}\left(S_{hx}^2\right)^{-1} Q_h\right)\left(\sum_{h=1}^{L}\left(S_{hx}^2\right)^{-1} Q_h \bar{x}_h^2\right) - \left(\sum_{h=1}^{L}\left(S_{hx}^2\right)^{-1} Q_h \bar{x}_h\right)^2}$$

2.1.3 Third Distance Function $\left(Z_3\right)$

Applying Lagrange multipliers method to minimize (1.12), presence of the constraints given in (1.5) and (1.6), we have the Lagrange function as

$$L_3 = \sum_{h=1}^{L} \frac{W_h}{Q_h}\left(\frac{\Omega_h^J}{W_h} - 1\right)^2 - 2\lambda_5\left(\sum_{h=1}^{L} \Omega_h^J - \sum_{h=1}^{L} W_h\right) - 2\lambda_6\left(\sum_{h=1}^{L} \Omega_h^J \bar{x}_h - \bar{X}\right) \tag{1.29}$$

where λ_5 and λ_6 are the Lagrange multipliers. To find Ω_h^J, (1.29) is partially differentiate with respect to Ω_h^J and equating to zero as:

$$\frac{\partial L_3}{\partial \Omega_h^J} = 2\sum_{h=1}^{L} \frac{W_h}{Q_h}\left(\frac{\Omega_h^J}{W_h} - 1\right) - 2\lambda_5 - 2\lambda_6 \bar{x}_h = 0$$

(1.30)

From (1.30), we get

$$\Omega_h^J = W_h + \lambda_5 W_h Q_h + \lambda_6 W_h Q_h \bar{x}_h = 0$$

(1.31)

Using (1.31), (1.5) and (1.6) then λ_5 and λ_6 are

$$\lambda_5 = -\frac{\left(\bar{X}_h - \sum_{h=1}^{L} W_h \bar{x}_h\right)\left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h\right)}{\left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h^2\right)\left(\sum_{h=1}^{L} W_h Q_h\right) - \left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h\right)^2}$$

(1.32)

$$\lambda_6 = \frac{\left(\bar{X}_h - \sum_{h=1}^{L} W_h \bar{x}_h\right)\left(\sum_{h=1}^{L} W_h Q_h\right)}{\left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h^2\right)\left(\sum_{h=1}^{L} W_h Q_h\right) - \left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h\right)^2}$$

(1.33)

Substituting (1.32) and (1.33) in (1.31), the new calibration weights can be written as

$$\Omega_h^J = W_h - \frac{\left(\bar{X}_h - \sum_{h=1}^{L} W_h \bar{x}_h\right)\left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h\right)}{\left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h^2\right)\left(\sum_{h=1}^{L} W_h Q_h\right) - \left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h\right)^2}\left(W_h Q_h\right) + \frac{\left(\bar{X}_h - \sum_{h=1}^{L} W_h \bar{x}_h\right)\left(\sum_{h=1}^{L} W_h Q_h\right)}{\left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h^2\right)\left(\sum_{h=1}^{L} W_h Q_h\right) - \left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h\right)^2}\left(W_h Q_h \bar{x}_h\right)$$

(1.34)

Substituting (1.34) in (1.3), the new estimator is

$$\bar{y}_{st}^J = \sum_{h=1}^{L} W_h \bar{y}_h + \hat{\beta}_j\left(\bar{X} - \sum_{h=1}^{L} W_h \bar{x}_h\right)$$

(1.35)

where

$$\hat{\beta}_j = \frac{\left(\sum_{h=1}^{L} W_h Q_h \bar{y}_h\right)\left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h\right) - \left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h \bar{y}_h\right)\left(\sum_{h=1}^{L} W_h Q_h\right)}{\left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h^2\right)\left(\sum_{h=1}^{L} W_h Q_h\right) - \left(\sum_{h=1}^{L} W_h Q_h \bar{x}_h\right)^2}$$

2.1.4 Fourth Distance Function $\left(Z_4\right)$

Applying Lagrange multipliers method to minimize (1.13), presence of the constraints given in (1.5) and (1.6), we have the Lagrange function as

$$L_4 = \frac{1}{2}\sum_{h=1}^{L} \frac{\left(\Omega_h^A - W_h\right)^2}{W_h Q_h} + \sum_{h=1}^{L}\sum_{h \neq h^l = 1}^{L}\left(\Omega_h^A - W_h\right)\left(W_{h^l}^{(1)} - W_{h^l}\right) - 2\lambda_7\left(\sum_{h=1}^{L} \Omega_h^A - \sum_{h=1}^{L} W_h\right) - 2\lambda_8\left(\sum_{h=1}^{L} \Omega_h^A \bar{x}_h - \bar{X}\right)$$

(1.36)

where λ_7 and λ_8 are the Lagrange multipliers. To find Ω_h^A, (1.36) is partially differentiate with respect to Ω_h^A and equating to zero as:

$$\Omega_h^A = W_h + \lambda_7\left(\frac{W_h Q_h}{1 - W_h Q_h}\right) + \lambda_8\left(\frac{W_h Q_h \bar{x}_h}{1 - W_h Q_h}\right) \quad (1.37)$$

Using (1.37) in (1.5) and (1.6) then λ_7 and λ_8 are

$$\lambda_7 = -\frac{\left(\bar{X}_h - \sum_{h=1}^{L} W_h \bar{x}_h\right)\left(\sum_{h=1}^{L}\left(\frac{W_h Q_h \bar{x}_h}{1 - W_h Q_h}\right)\right)}{\left(\sum_{h=1}^{L}\left(\frac{W_h Q_h}{1 - W_h Q_h}\right)\right)\left(\sum_{h=1}^{L}\left(\frac{W_h Q_h \bar{x}_h^2}{1 - W_h Q_h}\right)\right) - \left(\sum_{h=1}^{L}\left(\frac{W_h Q_h \bar{x}_h}{1 - W_h Q_h}\right)\right)^2} \quad (1.38)$$

$$\lambda_8 = \frac{\left(\bar{X}_h - \sum_{h=1}^{L} W_h \bar{x}_h\right)\left(\sum_{h=1}^{L}\left(\frac{W_h Q_h}{1 - W_h Q_h}\right)\right)}{\left(\sum_{h=1}^{L}\left(\frac{W_h Q_h}{1 - W_h Q_h}\right)\right)\left(\sum_{h=1}^{L}\left(\frac{W_h Q_h \bar{x}_h^2}{1 - W_h Q_h}\right)\right) - \left(\sum_{h=1}^{L}\left(\frac{W_h Q_h \bar{x}_h}{1 - W_h Q_h}\right)\right)^2}$$

(1.39)

Substituting (1.38) and (1.39) in (1.37), the new calibration weights can be written as

$$\Omega_h^A = W_h + \frac{\left(\bar{X}_h - \sum_{h=1}^{L} W_h \bar{x}_h\right)}{\left(\sum_{h=1}^{L}\left(\frac{W_h Q_h}{1 - W_h Q_h}\right)\right)\left(\sum_{h=1}^{L}\left(\frac{W_h Q_h \bar{x}_h^2}{1 - W_h Q_h}\right)\right) - \left(\sum_{h=1}^{L}\left(\frac{W_h Q_h \bar{x}_h}{1 - W_h Q_h}\right)\right)^2} \left[\left(\frac{W_h Q_h \bar{x}_h}{1 - W_h Q_h}\right)\left(\sum_{h=1}^{L}\left(\frac{W_h Q_h}{1 - W_h Q_h}\right)\right) - \left(\frac{W_h Q_h}{1 - W_h Q_h}\right)\left(\sum_{h=1}^{L}\left(\frac{W_h Q_h \bar{x}_h}{1 - W_h Q_h}\right)\right)\right]$$

(1.40)

Substituting (1.40) in (1.3), the new calibration estimator is

$$\bar{y}_{st}^A = \sum_{h=1}^{L} W_h \bar{y}_h + \hat{\beta}_a\left(\bar{X} - \sum_{h=1}^{L} W_h \bar{x}_h\right)$$

(1.41)

where

$$\hat{\beta}_a = \frac{\sum_{h=1}^{L}\left(\frac{W_hQ_h\bar{x}_h\bar{y}_h}{1-W_hQ_h}\right)\sum_{h=1}^{L}\left(\frac{W_hQ_h}{1-W_hQ_h}\right)-\sum_{h=1}^{L}\left(\frac{W_hQ_h\bar{x}_h}{1-W_hQ_h}\right)\sum_{h=1}^{L}\left(\frac{W_hQ_h\bar{y}_h}{1-W_hQ_h}\right)}{\sum_{h=1}^{L}\left(\frac{W_hQ_h}{1-W_hQ_h}\right)\sum_{h=1}^{L}\left(\frac{W_hQ_h\bar{x}_h^2}{1-W_hQ_h}\right)-\sum_{h=1}^{L}\left(\frac{W_hQ_h\bar{x}_h}{1-W_hQ_h}\right)^2}$$

2.1.5 Fifth Distance Function (Z_5)

Applying Lagrange multipliers method to minimize (1.14), presence of the constraints given in (1.5) and (1.6), we have the Lagrange function as

$$L_5 = \sum_{h=1}^{L}\frac{1}{Q_h}\left(\frac{\Omega_h^K}{W_h}-1\right)^2 - 2\lambda_9\left(\sum_{h=1}^{L}\Omega_h^K-\sum_{h=1}^{L}W_h\right)-2\lambda_{10}\left(\sum_{h=1}^{L}\Omega_h^K\bar{x}_h-\bar{X}\right) \quad (1.42)$$

where λ_9 and λ_{10} are the Lagrange multipliers. To find Ω_h^K, (1.42) is partially differentiate with respect to Ω_h^K and equating to zero as:

$$\frac{\partial L_5}{\partial \Omega_h^K} = 2\sum_{h=1}^{L}\frac{1}{Q_h}\left(\frac{\Omega_h^J}{W_h}-1\right)-2\lambda_9-2\lambda_{10}\bar{x}_h=0 \quad (1.43)$$

From (1.43), we have

$$\Omega_h^K = W_h+\lambda_9W_h^2Q_h+\lambda_{10}W_h^2Q_h\bar{x}_h=0 \quad (1.44)$$

Using (1.44), (1.15) and (1.16) then λ_9 and λ_{10} are

$$\lambda_9 = -\frac{\left(\bar{X}_h-\sum_{h=1}^{L}W_h\bar{x}_h\right)\left(\sum_{h=1}^{L}W_h^2Q_h\bar{x}_h\right)}{\left(\sum_{h=1}^{L}W_h^2Q_h\bar{x}_h^2\right)\left(\sum_{h=1}^{L}W_h^2Q_h\right)-\left(\sum_{h=1}^{L}W_h^2Q_h\bar{x}_h\right)^2} \quad (1.45)$$

$$\lambda_{10} = \frac{\left(\bar{X}_h-\sum_{h=1}^{L}W_h\bar{x}_h\right)\left(\sum_{h=1}^{L}W_h^2Q_h\right)}{\left(\sum_{h=1}^{L}W_h^2Q_h\bar{x}_h^2\right)\left(\sum_{h=1}^{L}W_h^2Q_h\right)-\left(\sum_{h=1}^{L}W_h^2Q_h\bar{x}_h\right)^2} \quad (1.46)$$

Substituting (1.45) and (1.46) in (1.44), the new calibration weights can be written as

$$\Omega_h^K = W_h - \frac{\left(\bar{X}_h-\sum_{h=1}^{L}W_h\bar{x}_h\right)\left(\sum_{h=1}^{L}W_h^2Q_h\bar{x}_h\right)}{\left(\sum_{h=1}^{L}W_h^2Q_h\bar{x}_h^2\right)\left(\sum_{h=1}^{L}W_h^2Q_h\right)-\left(\sum_{h=1}^{L}W_h^2Q_h\bar{x}_h\right)^2}\left(W_h^2Q_h\right) + \frac{\left(\bar{X}_h-\sum_{h=1}^{L}W_h\bar{x}_h\right)\left(\sum_{h=1}^{L}W_h^2Q_h\right)}{\left(\sum_{h=1}^{L}W_h^2Q_h\bar{x}_h^2\right)\left(\sum_{h=1}^{L}W_h^2Q_h\right)-\left(\sum_{h=1}^{L}W_h^2Q_h\bar{x}_h\right)^2}\left(W_h^2Q_h\bar{x}_h\right) \quad (1.47)$$

Substituting (1.47) in (1.3), a new calibration estimator is

$$\bar{y}_{st}^K = \sum_{h=1}^{L}W_h\bar{y}_h+\hat{\beta}_k\left(\bar{X}-\sum_{h=1}^{L}W_h\bar{x}_h\right) \quad (1.48)$$

where

$$\hat{\beta}_k = \frac{\left(\sum_{h=1}^{L}W_h^2Q_h\bar{y}_h\right)\left(\sum_{h=1}^{L}W_h^2Q_h\bar{x}_h\right)-\left(\sum_{h=1}^{L}W_h^2Q_h\bar{x}_h\bar{y}_h\right)\left(\sum_{h=1}^{L}W_h^2Q_h\right)}{\left(\sum_{h=1}^{L}W_h^2Q_h\bar{x}_h^2\right)\left(\sum_{h=1}^{L}W_h^2Q_h\right)-\left(\sum_{h=1}^{L}W_h^2Q_h\bar{x}_h\right)^2}$$

III. Simulation And Results

Simulation study is carried out to assess the performance and the efficiency of the distance functions of calibration estimators. A simulated study generating 2000 units were generated for study Populations stratified into 3 non-overlapping heterogeneous groups as 400, 600 and 1000 using function defined in Table 3.1. Samples of sizes 100, 250 and 400 were selected 20,000 times by method SRSWOR from each stratum respectively. Mean square error (MSE) of the considered estimators were computed as:

$$MSE\left(\bar{y}_{st}^*(s)\right) = \frac{1}{20000}\sum_{j=1}^{20000}\left[\bar{y}_{st}^*(s)-\bar{Y}\right]^2,\ \bar{y}_{st}^*(s)=\bar{y}_{st}^*,\bar{y}_{st}^M,\bar{y}_{st}^J,\bar{y}_{st}^A,\bar{y}_{st}^K \quad (1.49)$$

.

Table 3.1: Populations Used for Empirical Study

Population	Auxiliary variable x	Study variable y
I	$x_h \approx norm(\theta_h), \theta_1 = 5, \theta_2 = 6, \theta_3 = 4, h = 1,2,3$	$y_{hi} = \alpha_h x_{hi}^2 + \xi_{hi}$, $\alpha_{ih} = E(x_h)$, $\alpha = 1.0$, $\xi_h \approx N(0,1), h = 1,2,3$
II	$x_h \approx \exp(\theta_h), \theta_1 = 5, \theta_2 = 6, \theta_3 = 4, h = 1,2,3$	
III	$x_h \approx gamma(\theta_h, \eta_h), \theta_1 = 3, \eta_1 = 2, \theta_2 = 3, \eta_2 = 1, \theta_3 = 3, \eta_3 = 3,$	
IV	$x_h \approx \text{chisq}(\theta_h), \theta_1 = 5, \theta_2 = 6, \theta_3 = 4, h = 1,2,3$	

Different distributions used for simulation.

Table 3.2: Mean Square Error (MSE) of Estimators

Estimator	Population I	Population II	Population III	Population IV
$\bar{y}_{st}^{*}$	365656.5	3.946134	0.4039981	2.785017
$\bar{y}_{st}^{M}$	1950916	15.57206	1.759763	8.921438
$\bar{y}_{st}^{J}$	365656.5	3.946134	0.4039981	2.785017
$\bar{y}_{st}^{A}$	**222695.9**	**1.723568**	**0.3811151**	**1.191202**
$\bar{y}_{st}^{K}$	2403014	15.92707	1.843591	9.31367

Table 3.2 shows Mean Square Error of the estimators for Populations I, II, III, and IV. The result revealed that the calibrated estimator $\left(\bar{y}_{st}^{A}\right)$ of distance function Z_4 has a minimum MSE follow by distance functions Z_1 and Z_3 of calibrated estimators $\left(\bar{y}_{st}^{*}\right)$ and $\left(\bar{y}_{st}^{J}\right)$ (both have the same Mean Square Errors). This implies that the distance function Z_4 is proficient and more efficient than distance functions Z_1, Z_2, Z_3, and Z_5.

IV. Conclusion

In this study, new calibration weights are derived by minimization of five different distance functions subject to a pair of calibration constraints introduced by Alam et al. (2019) in stratified random sampling. A simulation study is carried out to show the performance of the distance measure. It is proved that the calibration estimator $\left(\bar{y}_{st}^{A}\right)$ using distance function Z_4 gives better estimate for the population mean of the study variable and more efficient than distance functions Z_1, Z_2, Z_3, and Z_5.

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