Theoretical Analysis of Topp-Leone Exponentiated Weibull Distribution as a Life Time Distribution

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Abstract **— This paper introduces a new lifetime model which is the generalization exponentiated exponential distribution using the Topp-Leone generated family of distributions proposed by Rezaei et al. The new distribution is called the Topp-leone Exponentiated Weibull (TLEW) distribution. Further, expressions for several probabilistic measures were provided, such as probability density function, hazard function, moments, quantile function, mean, variance and median, moment generating function, orders statistics etc. Inference is maximum likelihood based and tractability of model was shown by its application to a real data set.**

Keywords - *Moment generating function, Exponentiated Weibull distribution, Survival function, order statistics..*

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I. Introduction

Recently, a new generalization of the Weibull distribution, named Exponentiated Weibull (EW) distribution, as an alternative distribution to the exponential, Weibull and gamma was proposed by [6]. The cumulative distribution function (cdf) of the EW distribution is given by

$$F(x;\alpha,\beta,\upsilon)=\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{\upsilon} \qquad (1)$$

And the corresponding probability density function (pdf) is

$$f(x;\alpha,\beta,\upsilon)=\frac{\alpha\upsilon}{\beta^{\alpha}}x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{\upsilon-1} \qquad (2)$$

Where the parameter $\alpha, \upsilon > 0$ controls the shape of the distribution $\beta > 0$ and is the scale parameter that depends on α.

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II. Topp-Leone Exponentiated Weibull Distribution

We shall refer to the new distribution using (1) and (2) as the Topp-Leone Exponentiated Weibull distribution TLEW using the Topp-Leone generated (TLG) family of distributions which was introduced by Rezaei, S et al. The pdf and cdf of the TLG family of distributions are given by

$$G(x;\zeta)=F(x;\zeta)^{\theta}(2-F(x;\zeta))^{\theta}, \qquad (3)$$

And

$$G(x;\zeta)=2\theta f(x;\zeta)\big(1-F(x;\zeta)\big)F(x;\zeta)^{\theta-1}\big(2-F(x;\zeta)\big)^{\theta-1} \qquad (4)$$

where $\theta > 0$ is the shape parameter and ζ is the parameter vector of the baseline distribution G.

By inserting (1) and (2) into (3) and (4), we can write the pdf and cdf of the TLEW distribution as

$$g(x)=2\frac{\alpha\theta\upsilon}{\beta^{\alpha}}x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{\upsilon}\left\{1-\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{\upsilon}\right\}\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{\upsilon(\theta-1)}\left[2-\left(1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)^{\upsilon}\right]^{\theta-1} \qquad (5)$$

And

$$G(x)=\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{\upsilon\theta}\left[2-\left(1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)^{\upsilon}\right]^{\theta} \qquad (6)$$

The additional shape parameter θ is to induce more flexibility into the model and gives the hazard function a wider area of applications.

The graph of the TLEW distribution is given below, where $a=\alpha, b=\beta, b1=\upsilon, c=\theta$

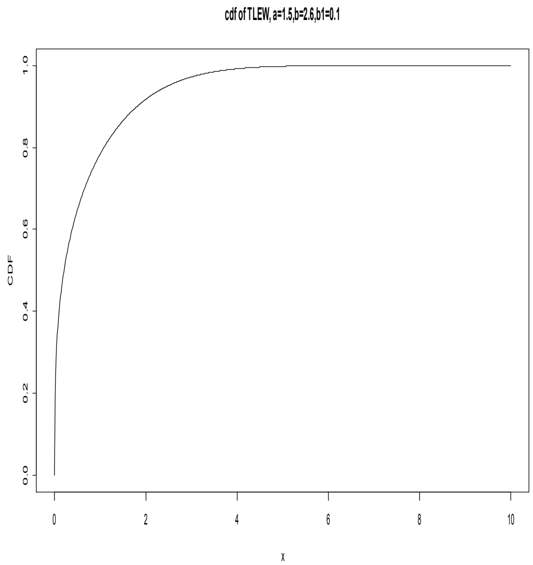

Fig. 1.0: The graph of the cdf of TLEW distribution function

The figure 1.0 indicates that the TLEW distribution is a proper or true pdf.

The equation (3) and (4) can be represented as a mixture distribution as given below:

$$G(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{iv}(-1)^{i+j}\binom{\theta}{i}\binom{iv}{j}2^{\theta-i}e^{-j\left(\frac{x}{\beta}\right)^{\alpha}} \quad (7)$$

and

$$g(x) = 2\frac{\alpha\theta v}{\beta^{\alpha}}x^{\alpha-1}\sum_{j,k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{vr-1} lZ_{j,k,l,m}\, e^{-\left(\frac{x}{\beta}\right)^{\alpha}(1+m)} \quad (8)$$

Where

$$Z_{j,k,l,m} = \binom{\theta}{j}\binom{\theta+j}{k}\binom{k}{l}\binom{vl-1}{m}(-1)^{j+k+l+m}2^{\theta-j} \quad (9)$$

The graph below depicts the graph of the pdf of TLEW distribution function for various values of the parameters.

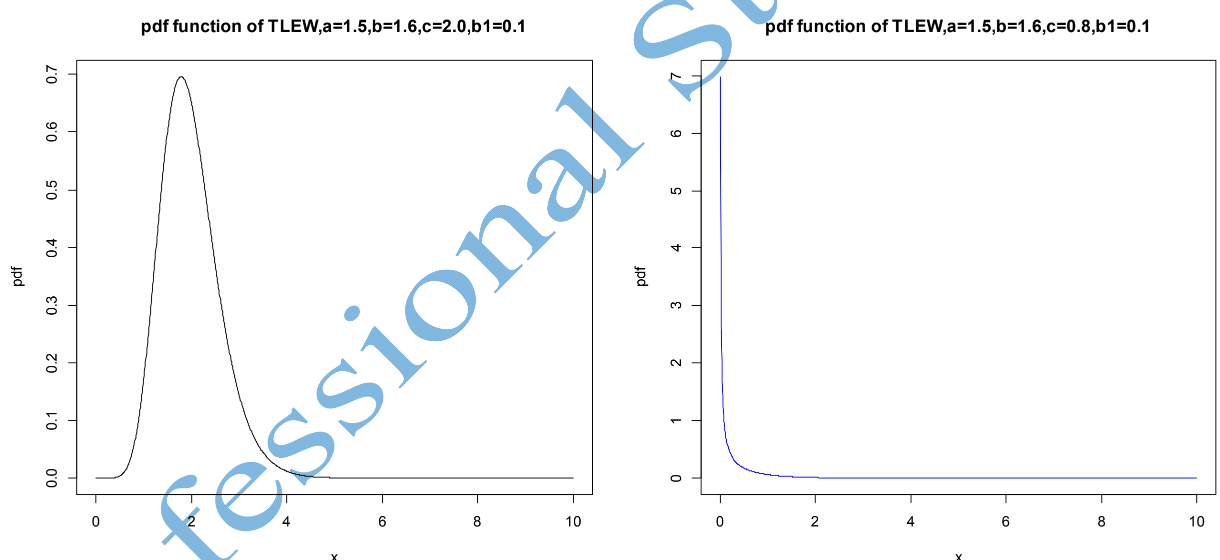

Fig. 2.0 The graph of the pdf of TLEW distribution

The following can be deduced from the graphical illustration above on the behaviour of the density function:

a. The graph of the pdf of TLEW distribution is unimodal and right tailed when $\theta > 1$ *and* $v > 1$.
b. The graph of the pdf of TLEW distribution is a decreasing function when $\theta < 1$ *and* $v < 1$.

III. Hazard Function

The hazard rate function also known as instantaneous failure rate defined by

$$h(x) = \frac{f(x)}{1-F(x)} \quad (10)$$

Is an important quantity characterizing life phenomenon. It can be loosely interpreted as the conditional probability of failure, given it has survived to the time *t:* The hazard rate function for a TLEW distribution is given by

$$h(x) = \frac{2\frac{\alpha\theta\upsilon}{\beta^{\alpha}}x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{\upsilon}\left\{1-\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{\upsilon}\right\}\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{\upsilon(\theta-1)}\left[2-\left(1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)^{\upsilon}\right]^{\theta-1}}{1-\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{\upsilon\theta}\left[2-\left(1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)^{\upsilon}\right]^{\theta}} \quad (11)$$

The graph of the hazard function is drawn below for various values of the parameters:

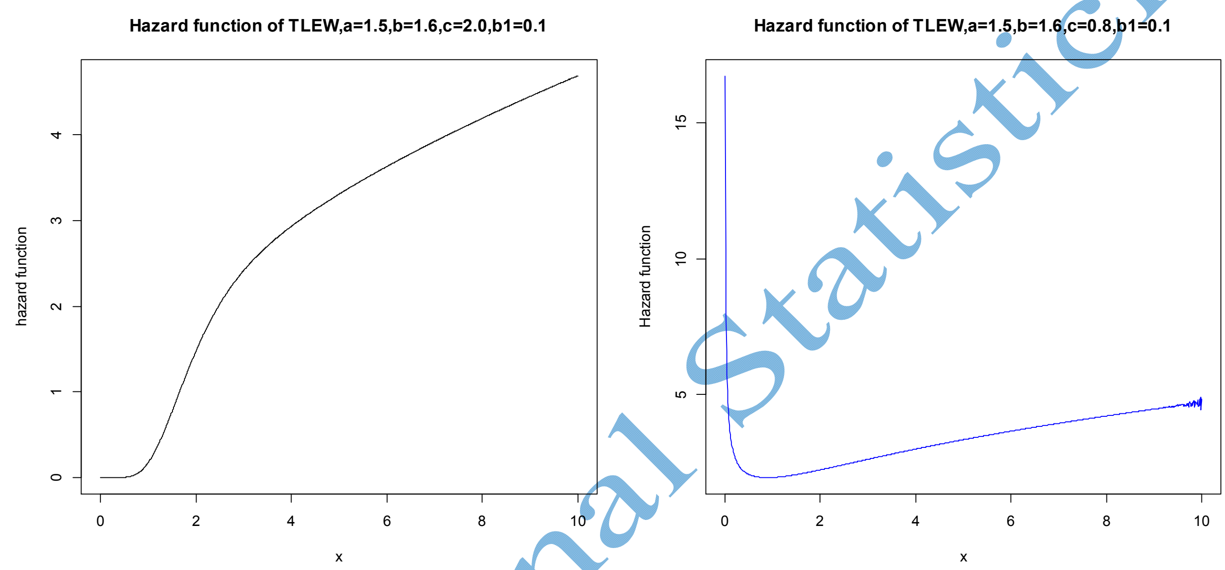

Fig. 3.0: Graphs of hazard function for various values of the parameters.

The following can be deduced from the graphical illustration above on the behaviour of the density function:

a. The graph of the Hazard function of TLEW distribution has an increasing Hazard function when $\theta < 1$ *and* $\upsilon < 1$.

b. The graph of the Hazard function of TLEW distribution exhibits a non-monotone failure rate when $\theta > 1$ *and* $\upsilon > 1$.

IV. Reliability Function

The survival function, also known as the reliability functions in engineering, in fact it is define as the conditional probability that the system will survive beyond a specified time. The TLEW distribution can be a useful model to characterize failure time of a given system because of the analytical structure. The reliability function $R(x)$, which is the probability of an item not failing prior to time t, is defined by $R(x) = 1-G(x)$. The reliability function define as, $R(x) = 1 - G(x)$, for TLEW distribution is given by:

$$R(t) = 1-\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{\upsilon\theta}\left[2-\left(1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)^{\upsilon}\right]^{\theta} \quad (12)$$

The graph drawn below depicts the graph of TLEW distribution for various values of the parameters.

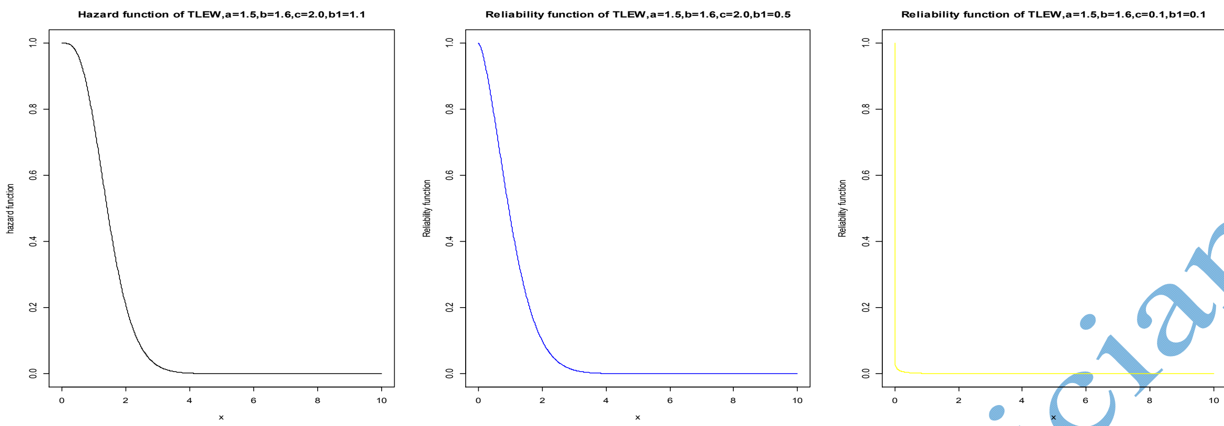

Fig. 4.0 The graph of the reliability function

V. Moments, Mean, Variance, Median of TLEW distribution

In this section we shall present the moments TLEW distribution. The p^{th} order moments, for TLEW distribution can be obtained as follows for a random variable *X*,

$$E(X)^p = \int_{-\infty}^{\infty} x^p f(x)dx \qquad (13)$$

Putting eq. (8) in eq. (13), we have

$$E(X)^k = 2\frac{\alpha\theta\upsilon}{\beta^\alpha}\sum_{j,k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{\upsilon r-1} lZ_{j,k,l,m}\int_{-\infty}^{\infty} x^{\alpha+p-1} e^{-\left(\frac{x}{\beta}\right)^\alpha (1+m)} \qquad (14)$$

Solving equation (14) by letting $\left(\frac{x}{\beta}\right)^\alpha (1+m) = w$

Finally we have

$$E(X)^p = \mu_p = 2\frac{\alpha\theta\upsilon\beta^p}{(1+m)^{\frac{\alpha+k}{\alpha}}}\sum_{j,k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{\upsilon r-1} lZ_{j,k,l,m}\Gamma\left\{\frac{p}{\alpha}+1,\left(\frac{t}{\beta}\right)^\alpha(1+m)\right\} \qquad (15)$$

The first moment gives the mean of the distribution when $p = 1$

$$\mu_1 = 2\frac{\alpha\theta\upsilon\beta}{(1+m)^{\frac{\alpha+1}{\alpha}}}\sum_{j,k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{\upsilon r-1} lZ_{j,k,l,m}\Gamma\left\{\frac{1}{\alpha}+1,\left(\frac{t}{\beta}\right)^\alpha(1+m)\right\} \qquad (16)$$

Second moment, $p = 2$

$$\mu_2 = 2\frac{\alpha\theta\upsilon\beta^2}{(1+m)^{\frac{\alpha+2}{\alpha}}}\sum_{j,k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{\upsilon r-1} lZ_{j,k,l,m}\Gamma\left\{\frac{2}{\alpha}+1,\left(\frac{t}{\beta}\right)^\alpha(1+m)\right\} \qquad (17)$$

From $V(x) = \mu_2 - (\mu_1)^2$

Therefore

$$V(x) = 2\frac{\alpha\theta\upsilon\beta^2}{(1+m)^{\frac{\alpha+2}{\alpha}}}\sum_{j,k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{\upsilon r-1} lZ_{j,k,l,m}\Gamma\left\{\frac{2}{\alpha}+1,\left(\frac{t}{\beta}\right)^\alpha(1+m)\right\} - \left(2\frac{\alpha\theta\upsilon\beta}{(1+m)^{\frac{\alpha+1}{\alpha}}}\sum_{j,k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{\upsilon r-1} lZ_{j,k,l,m}\Gamma\left\{\frac{1}{\alpha}+1,\left(\frac{t}{\beta}\right)^\alpha(1+m)\right\}\right)^2 \qquad (18)$$

The skewness of the distribution can be obtained when we consider μ_3 and μ_4

VI. Moment Generating Function of TLEW Distributions

The moment generating function of a random variable x is defined by

$$M_t(x) = \int_{-\infty}^{\infty} e^{tx} f(x)dx \qquad (19)$$

The above expression can further be simplify as

$$M_t(x) = \sum_{p=0}^{\infty}\frac{t^p}{p!}\int_{-\infty}^{\infty} x^p f(x)dx \qquad (20)$$

Since,

$$e^{tx} = \sum_{r=0}^{\infty} \frac{t^p x^p}{p!} \qquad (21)$$

Inserting eq. (15) in eq. (20) we have

$$M_t(x) = 2\frac{\alpha\theta v\beta^p}{(1+m)^{\frac{\alpha+2}{\alpha}}} \sum_{p=0}^{\infty}\sum_{j,k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{vr-1} \frac{t^p}{p!} l Z_{j,k,l,m} \Gamma\left\{\frac{1}{\alpha} + 1, \left(\frac{t}{\beta}\right)^{\alpha}(1+m)\right\} \qquad (22)$$

VII. Estimation And Statistical Inference

Let $x_1, x_2, \dots, x_n$ be random variable distributed according to (8) the likelihood function of a vector of parameters given as $\Omega(\alpha, \beta, \theta, v)$. the log likelihood function is given as

$$\begin{aligned} l(\Omega) = {} & nlog\alpha + nlog\theta + nlogv - n\alpha log\beta \\ & + (\alpha - 1)\sum_{i=1}^{n} logx_i \\ & - \alpha\sum_{i=1}^{n} log\left(\frac{x_i}{\beta}\right) + v\sum_{i=1}^{n} log\left[1 - e^{-\left(\frac{x_i}{\beta}\right)^{\alpha}}\right] \\ & \sum_{i=1}^{n} log\left\{1 - \left[1 - e^{-\left(\frac{x_i}{\beta}\right)^{\alpha}}\right]^{v}\right\} + v(\theta - 1)\sum_{i=1}^{n} log\left[1 - e^{-\left(\frac{x_i}{\beta}\right)^{\alpha}}\right] \\ & + (\theta \\ & - 1)\sum_{i=1}^{n} log\left[2 - \left(1 - e^{-\left(\frac{x_i}{\beta}\right)^{\alpha}}\right)^{v}\right] \end{aligned} \qquad (23)$$

VIII. Applications

We present an application based on the real data set to show the flexibility of the TLEW distribution. We compare TLEW distribution with the Topp-leone Weibull (TLW) distribution and the 2-parameter Weibull distribution using the estimated log-likelihood value, Kolmogorov-Smirnov (K-S) statistics, Anderson Darling statistic (A^*), crammer Von Misses (V), Akaike information criterion (AIC), Consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQIC). The real data set represents the remission times (in months) of a random sample of 128 bladder cancer patients [7]. This data is known to have a unimodal hazard rate function shaped in the literature.

The Exploratory data analysis is given in table 1. The results of this application to the real data set are listed in Table 2 and Table 3. These results indicates that the TLEW distribution has the lowest AIC, CAIC, BIC, HQIC, K-S, A^*, and W values and has the biggest estimated log-likelihood (l) and p-value of the K-S statistics among all the fitted models. Hence, it could be chosen as the best model under these criteria.

Table 1.0 Descriptive Statistics on Cancer remission time.

Min	Q_1	Median	$mean$	Q_3	Max	$kurtosis$	Skewness	$Variance$
0.0047	0.3723	0.4734	0.4810	0.6405	0.9774	16.1537	3.3256	110.425

Table2 MLEs(standard error in parenthesis) and the statistics $l(\hat{\theta})$, AIC, BIC and $HQIC$

$Model$	$Estimates$				$l(\hat{\theta})$	AIC	BIC	HQIC	CAIC
$TLEW$ $(\alpha, \beta, v, \theta)$	3.8608 (0.4705)	0.71088 (0.0272)	0.0297 (0.0133)	14.9858 (6.3125)	-128.761	-249.522	-229.891	-242.061	-249.482
TLW (α, β, θ)	4.6847 (0.5353)	0.8393 (0.0147)	- -	0.3571 (0.0147)	-125.618	-245.237	-230.513	-239.641	-245.213
W (α, β)	2.3123 (0.0593)	0.5417 (0.0078)	- -		-104.717	-205.433	-195.618	-201.702	-205.421

Table 3.0 : Table of Test for Goodness-of-Fit

$Model$	A^*	W	$K-S$	P
$TLEW$ $(\alpha, \beta, v, \theta)$	0.6354	0.07031	0.01929	0.8506
TLW (α, β, θ)	0.9085	0.1076	0.0242	0.6004
W (α, β)	2.2392	0.3204	0.0421	0.0574

VIII. Conclusion

The well-known three-parameter exponentiated Weibull distribution was extended by adding one extra shape parameters, using Topp-leone generator, thus having a distribution with a broader class of hazard rate functions. This is achieved by taking (1) as the baseline cumulative distribution of the generalized class of Exponentiated Weibull distributions defined by Mudholkar, G. S et al. [6]. A detailed study on the mathematical properties of the new distribution obtained. The new model has a sub-models which include exponentiated Weibull, Weibull and exponential distribution. We obtain the moment generating function, ordinary moments, order statistics hazard rate function and the Reliability function. The model parameters were estimated using maximum likelihood and the observed information matrix is derived. An application to a real data set shows that the fit of the new model is superior to the fits of its main sub-models.

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