Theoritical Analysis of Odd Generalized Exponential Inverse-Exponential Distribution

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***Abstract*—There exist many problems in real life where observed data do not follow any of the well-known probability distributions. With this, there is need to propose probability models that better capture the behavior of some real-life phenomenon. In this paper, we propose a new lifetime distribution, called the Odd Generalized Exponential Inverse-Exponential Distribution (OGE-IED) based on the odd generalized exponential generator proposed in earlier study. The statistical properties of the new distribution are studied along with its reliability functions and limiting behavior. Method of maximum likelihood was used to estimate the parameters of the distribution.**

Keywords: *Generalized exponential, Moments, Moment generating function, Maximum likelihood.*

I. Introduction

Despite several attempts by various researchers to develop new probability models, there still remain many important real-life problems where the observed data do not follow any of the classical probability models [1]. To tackle this challenge, there is need to propose other probability models that can better capture the behavior of these datasets. This idea started with defining different mathematical functional forms, and then inducing location, scale or shape parameters [2]. Induction of new shape parameter(s) into a model introduces it into a larger family and can provide significantly skewed and heavy-tailed new distributions. The practice of inducing one or more new shape parameter(s) has been proved useful in exploring tail properties and also for improving the goodness-of-fit of the proposed generator family [3].

A one-parameter Inverse Exponential distribution introduced by [4] has an inverted bathtub failure rate and can be compared competitively with Exponential distribution. It is used in modelling lifetime data. Recently, several generalization of inverse-exponential distribution were obtained, including Kumaraswamy inverse-exponential distribution [5], Exponentiated generalized inverse-exponential distribution [6], Transmuted inverse-exponential distribution [7], Generalized inverse-exponential distribution [8] and Modified generalized inverse exponential distribution by [9]. On the other hand, [10] proposed a generalization of the exponential distribution called Generalized Exponential (GE) distribution.

Recently a new generator called the odd generalized exponential-G (OGE-G) was proposed by [11] and each of OGE-Weibull (OGE-W) model, the OGE-Fréchet (OGE-Fr) model and the OGE-Normal (OGE-N) models were thoroughly studied. This method is more flexible because the hazard rate shapes could be increasing, decreasing, bathtub and sometimes upside down bathtub. The main aim for employing the technique advanced by the OGE generator is to propose more flexible and highly skewed distributions for modeling real-life datasets [11]. Thereafter, many OGE-G distributions have appeared in the literature, including, OGE-Gompertz distribution [12].

II. Research Methodology

In this article, we present a new distribution which is a hybrid of the generalized exponential distribution and inverse-exponential distribution called Odd Generalized Exponential Inverse-Exponential Distribution (OGE-IED) using a new OGE family generator proposed by [11].

A random variable $X$is said to follow a generalized exponential (GE) distribution with parameters $\alpha > 0$ and $\beta > 0$, if its cumulative distribution function (CDF) and probability density function (pdf) are respectively given by:

$$F(x) = (1 - e^{-\alpha x})^{\beta} \quad (1)$$

$$f(x) = \alpha\beta e^{-\alpha x}(1 - e^{-\alpha x})^{\beta-1} \quad (2)$$

On the other hand, a random variable X that follows inverse-exponential distribution also known as inverted exponential distribution has its pdf and CDF respectively given by:

$$F(x) = e^{\frac{-\alpha}{x}} \qquad x, \alpha > 0 \quad (3)$$

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$$f(x) = \frac{\alpha}{x^2} e^{\frac{-\alpha}{x}} \tag{4}$$

The CDF of the OGE family defined by Tahir *et al.*, (2015) is obtained by replacing the variable $x$in equation (1) with $G(x,\omega)/\bar{G}(x,\omega)$to get

$$F(x) = \left(1 - e^{-\alpha\frac{G(x,\omega)}{\bar{G}(x,\omega)}}\right)^{\beta} \tag{5}$$

Where$\alpha > 0, \beta > 0$are the two additional parameters, $G(x,\omega)$is the CDF of any univariate continuous distribution defined on parameter vector ω, and $\bar{G}(x,\omega) = 1 - G(x,\omega)$is the survival function of $G(x,\omega)$.

A. The pdf and CDF of the OGE-IED

In this section, we define new three parameter distribution called odd generalized exponential inverse-exponential distribution $(OGE - IED)$ having parameters$\alpha > 0, \beta > 0$ and λ denoted as $OGE - IED(\boldsymbol{\varpi})$, where $\boldsymbol{\varpi}$ is a parameter vector defined by $\boldsymbol{\varpi} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\lambda})'$. A random variable X is said to follow $OGE - IED(\boldsymbol{\varpi})$, if its pdf and CDF are given as follows:

$$F(x) = \left(1 - e^{-\alpha\left(\frac{e^{\frac{-\alpha}{x}}}{1-e^{\frac{-\alpha}{x}}}\right)}\right)^{\beta} = \left(1 - e^{-\alpha\left(e^{\frac{\alpha}{x}}-1\right)^{-1}}\right)^{\beta} \tag{6}$$

$$f(x) = \frac{\alpha\beta\lambda e^{\frac{-\alpha}{x}}}{x^2\left(1-e^{\frac{-\alpha}{x}}\right)^2} e^{-\alpha\left(\frac{e^{\frac{-\alpha}{x}}}{1-e^{\frac{-\alpha}{x}}}\right)} \left(1 - e^{-\alpha\left(\frac{e^{\frac{-\alpha}{x}}}{1-e^{\frac{-\alpha}{x}}}\right)}\right)^{\beta-1} \tag{7}$$

Equation (7) can further be written as:

$$f(x) = \frac{\alpha\beta\lambda e^{\frac{-\alpha}{x}}}{x^2\left(1-e^{\frac{-\alpha}{x}}\right)^2} e^{-\alpha\left(e^{\frac{\alpha}{x}}-1\right)^{-1}} \left(1 - e^{-\alpha\left(e^{\frac{\alpha}{x}}-1\right)^{-1}}\right)^{\beta-1} \tag{8}$$

Special case:

For $\beta = 1$ in Equation (6) the proposed distribution will reduced to Odd Exponential Inverse-Exponential distribution.

The survival and hazard function of an OGE-IED distributed random variable X are respectively given by:

$$S_{OGE-IE}(x) = 1 - \left(1 - e^{-\alpha\left(e^{\frac{\alpha}{x}}-1\right)^{-1}}\right)^{\beta} \tag{9}$$

$$H_{OGE-IE}(x) = \frac{\frac{\alpha\beta\lambda e^{\frac{-\alpha}{x}}}{x^2\left(1-e^{\frac{-\alpha}{x}}\right)^2} e^{-\alpha\left(e^{\frac{\alpha}{x}}-1\right)^{-1}} \left(1 - e^{-\alpha\left(e^{\frac{\alpha}{x}}-1\right)^{-1}}\right)^{\beta-1}}{1 - \left(1 - e^{-\alpha\left(e^{\frac{\alpha}{x}}-1\right)^{-1}}\right)^{\beta}} \tag{10}$$

B. Statistical Properties

Asymptotic Behavior of the OGE-IED

Here, we tried to observe the behavior of the proposed model given in equation (7) as $x \to 0$ and as $x \to \infty$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\alpha\beta\lambda e^{\frac{-\alpha}{x}}}{x^2\left(1-e^{\frac{-\alpha}{x}}\right)^2}$$

$$\times e^{-\alpha\left(e^{\frac{\alpha}{x}}-1\right)^{-1}} \left(1 - e^{-\alpha\left(e^{\frac{\alpha}{x}}-1\right)^{-1}}\right)^{\beta-1} = 0 \tag{11}$$

And

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\alpha\beta\lambda e^{\frac{-\alpha}{x}}}{x^2\left(1 - e^{\frac{-\alpha}{x}}\right)^2}$$

$$\times e^{-\alpha\left(e^{\frac{\alpha}{x}}-1\right)^{-1}} \left(1 - e^{-\alpha\left(e^{\frac{\alpha}{x}}-1\right)^{-1}}\right)^{\beta-1} = 0 \tag{12}$$

Hence, OGE-IED has a unimodal.

Moments

Theorem: The r[th] moment of the random variable $X \sim OGE - IED(\boldsymbol{\varpi})$ is given by;

$$\mu'_r = \beta\Gamma(1-r) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{\beta-1}{i}\binom{-(j+2)}{k}$$

$$\times \frac{(j+k+1)^{r-1}(-1)^{i+j+k}\alpha^{j+1}(i+1)^j}{j!} \tag{13}$$

Proof: The r[th] moment of a random variable $X$with pdf $f(x)$ is defined by

$$\mu'_r = \int_0^{\infty} x^r f(x)dx \tag{14}$$

Substituting equation (8) into equation (14), we obtain

$$\mu'_r = \int_0^{\infty} x^r \frac{\alpha\beta\lambda e^{\frac{-\alpha}{x}}}{x^2\left(1-e^{\frac{-\alpha}{x}}\right)^2}$$

$$\times e^{-\alpha\left(e^{\frac{\alpha}{x}}-1\right)^{-1}} \left(1 - e^{-\alpha\left(e^{\frac{\alpha}{x}}-1\right)^{-1}}\right)^{\beta-1} dx \tag{15}$$

Applying binomial expansion on $\left(1 - e^{-\alpha\left(e^{\frac{\alpha}{x}}-1\right)^{-1}}\right)^{\beta-1}$, we obtain

$$\left(1 - e^{-\alpha\left(e^{\frac{\alpha}{x}}-1\right)^{-1}}\right)^{\beta-1} =$$

$$\sum_{i=0}^{\infty} \binom{\beta-1}{i} (-1)^i e^{-i\alpha\left(e^{\frac{\alpha}{x}}-1\right)^{-1}} \tag{16}$$

Substituting equation (16) into equation (15), we have

$$\mu_r' = \sum_{i=0}^{\infty}\binom{\beta-1}{i}(-1)^i\int_0^\infty x^r \frac{\alpha\beta\lambda e^{\frac{-\alpha}{x}}}{x^2\left(1-e^{\frac{-\alpha}{x}}\right)^2}$$

$$\times e^{-\alpha(i+1)\left(e^{\frac{\alpha}{x}}-1\right)^{-1}} dx \qquad (17)$$

Using series expansion of $e^{-\alpha(i+1)\left(\frac{e^{\frac{-\alpha}{x}}}{1-e^{\frac{-\alpha}{x}}}\right)} = e^{-\alpha(i+1)\left(e^{\frac{\alpha}{x}}-1\right)^{-1}}$, we obtain

$$\mu_r' = \beta\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\beta-1}{i}\frac{(-1)^{i+j}\alpha^{j+1}(i+1)^j}{j!}\int_0^\infty x^{r-2}$$

$$\times e^{\frac{-\lambda(j+1)}{x}}\left(1-e^{\frac{-\alpha}{x}}\right)^{-(j+2)} dx \qquad (18)$$

Using binomial expansion of $\left(1-e^{\frac{-\alpha}{x}}\right)^{-(j+2)}$, we obtain

$$\mu_r' = \beta\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k}^{\infty}\binom{\beta-1}{i}\binom{-(j+2)}{k}$$

$$\times \frac{(-1)^{i+j+k}\alpha^{j+1}(i+1)^j}{j!}\int_0^\infty x^{r-2}e^{\frac{-\lambda(j+1)}{x}}dx \qquad (19)$$

Letting $m = \frac{\lambda(j+k+1)}{x}$, we have

$$\mu_r' = \beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k}^{\infty}\binom{\beta-1}{i}\binom{-(j+2)}{k}$$

$$\times \frac{(j+k+1)^{r-1}(-1)^{i+j+k}\alpha^{j+1}\lambda^r(i+1)^j}{j!}\int_0^\infty m^{-r}e^{-m}dm \qquad (20)$$

Where,

$$\int_0^\infty m^{-r}e^{-m}dm = \Gamma(1-r) \qquad (21)$$

Is an incomplete gamma function.

Hence, the moment of the OGE-IED is given as

$$\mu_r' = \beta\Gamma(1-r)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\beta-1}{i}\binom{-(j+2)}{k}$$

$$\times \frac{(j+k+1)^{r-1}(-1)^{i+j+k}\alpha^{j+1}(i+1)^j}{j!} \qquad (22)$$

This complete the proof.

Moment Generating Function (MGF)

The moment generating function of the random variable X that follows OGE-IED having probability density function, $f(x)$ given in equation (7) is obtained by:

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx}f(x)dx \qquad (23)$$

Theorem: the MGF of the random variable $X \sim OGE-IED(\varpi)$ is given by;

$$M_x(t) = \beta\Gamma(1-m)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{\beta-1}{i}\binom{-(j+2)}{k}$$

$$\times \frac{(j+k+1)^{l-1}(-1)^{i+j+k}t^l\lambda^l\alpha^{j+1}(i+1)^j}{j!l!} \qquad (24)$$

Proof: The mgf of a random variable X with pdf $f(x)$ is defined by

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx}f(x)dx \qquad (25)$$

This can be derived by replacing X^r by e^{tx} in equation (19). That is

$$M_x(t) = \beta\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k}^{\infty}\binom{\beta-1}{i}\binom{-(j+2)}{k}\frac{(-1)^{i+j+k}\alpha^{j+1}(i+1)^j}{j!}$$

$$\times \int_0^\infty e^{tx}x^{-2}e^{\frac{-\lambda(j+k+1)}{x}}dx \qquad (26)$$

Using series expansion, $e^{tx} = \sum_{l=0}^{\infty}\frac{t^l x^l}{l!}$, we have

$$M_x(t) = \beta\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k}^{\infty}\binom{\beta-1}{i}\binom{-(j+2)}{k}\frac{t^l(-1)^{i+j+k}\alpha^{j+1}(i+1)^j}{j!\,l!}$$

$$\times \int_0^\infty x^{l-2}e^{\frac{-\lambda(j+k+1)}{x}}dx \qquad (27)$$

Letting $m = \frac{\lambda(j+k+1)}{x}$ and substituted appropriately, we obtain

$$M_x(t) = \beta\Gamma(1-m)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k}^{\infty}\binom{\beta-1}{i}\binom{-(j+2)}{k}\frac{t^l\lambda^l(-1)^{i+j+k}\alpha^{j+1}(i+1)^j}{j!l!} \qquad (28)$$

where,

$$\Gamma(1-m) = \int_0^\infty m^{-l}e^{-m}dm$$

This complete the proof.

III. Parameter Estimation Of OGE-IED

We make use of the method of maximum likelihood estimation (MLE) to estimate the parameters of the OGE-IE distribution. Let $X = (X_1, X_2, \ldots, X_n)'$ be a random sample of size $\boldsymbol{n}$ from the OGE-IED with parameter vector $\boldsymbol{\varpi} = (\alpha, \beta, \lambda)'$. Then the log-likelihood function for $\boldsymbol{\varpi}$ is given by;

$$l(\boldsymbol{\varpi}) = logL(X_1, X_2, \ldots, X_n / \boldsymbol{\varpi}) = log\prod_{i=1}^{n} f(x_i; \boldsymbol{\varpi})$$

$$= nln\alpha + nln\beta + nln\lambda - \lambda\sum_{i=1}^{n}\frac{1}{x_i} + 2\sum_{i=1}^{n}\frac{1}{x_i}$$

$$-2\sum_{i=1}^{n}ln\left(1-e^{\frac{-\lambda}{x}}\right) - \alpha\sum_{i=1}^{n}\left(1-e^{\frac{-\lambda}{x}}\right)$$

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$$+(\beta-1)\sum_{i=1}^{n} ln\left(1-e^{\alpha\left(e^{\frac{\lambda}{x}-1}\right)^{-1}}\right) \qquad (29)$$

Hence, the maximum likelihood estimate $\boldsymbol{\hat{\varpi}} = \left(\hat{\alpha},\hat{\beta},\hat{\lambda}\right)'$ of $\boldsymbol{\varpi} = (\alpha,\beta,\lambda)'$which maximize (29) must satisfy the following normal equations.

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\left(1-e^{\frac{-\lambda}{x_i}}\right)$$

$$+(\beta-1)\sum_{i=1}^{n}\frac{\left(e^{\frac{\lambda}{x_i}-1}\right)^{-1} e^{-\alpha\left(e^{\frac{\lambda}{x_i}-1}\right)^{-1}}}{\left(1-e^{-\alpha\left(e^{\frac{\lambda}{x_i}-1}\right)^{-1}}\right)} \qquad (30)$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} ln\left(1-e^{-\alpha\left(e^{\frac{\lambda}{x}-1}\right)^{-1}}\right) \qquad (31)$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}\frac{1}{x_i} - 2\sum_{i=1}^{n}\frac{e^{\frac{-\lambda}{x_i}}}{x_i\left(1-e^{\frac{-\lambda}{x_i}}\right)}$$

$$-\alpha\sum_{i=1}^{n}\frac{e^{\frac{-\lambda}{x}}}{x_i} - \alpha(\beta-1)\sum_{i=1}^{n}\frac{e^{\frac{-\lambda}{x_i}}\left(e^{\frac{\lambda}{x_i}}-1\right)^{-2} e^{-\alpha\left(e^{\frac{\lambda}{x_i}-1}\right)^{-1}}}{x_i\left(1-e^{-\alpha\left(e^{\frac{\lambda}{x}-1}\right)^{-1}}\right)} \qquad (32)$$

Hence, (MLE) $\boldsymbol{\hat{\varpi}} = \left(\hat{\alpha},\hat{\beta},\hat{\lambda}\right)'$ of $\boldsymbol{\varpi} = (\alpha,\beta,\lambda)'$is obtained by solving this nonlinear system of equations. These equations cannot be solved analytically and statistical software can be used to solve the equations numerically.

The R statistical package (www.cran.org)_was employed in this work and R codes were written to evaluate all the theoretical developments including the determination of the parameters of the new proposed distributions as provided above.

For clarity, the graphs of the probability density function, cumulative density function, survival function and hazard function of the proposed OGE-IED at varying values of the parameters are presented in the Appendix.

IV. Conclusion

In this study, we have developed a new generalization of the inverse-exponential distribution called the Odd Generalized Exponential Inverse-Exponential distribution (OGE-IED). The subject distribution is generated by using the odd generalized exponential generator (OGE-G) and taking the Inverse-exponential distribution as the base distribution. Some statistical properties of the proposed distribution were obtained. Further, mathematical expressions for some reliability (comprising survival and hazard) functions were also determined.

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Appendix

Graphs of the probability density function, cumulative density function, survival function and hazard function of the proposed OGE-IED at varying values of the parameters.

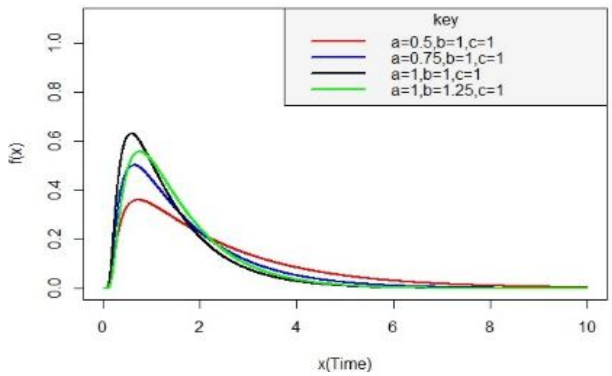

Figure 1: pdf plot of OGE-IED

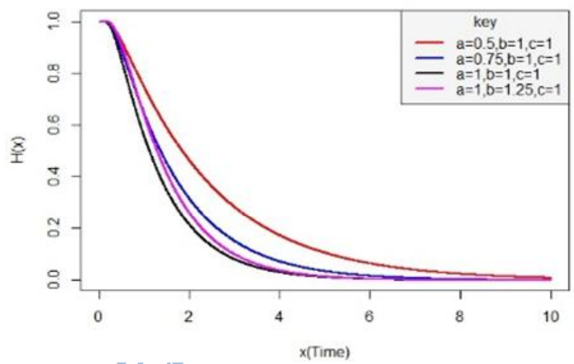

Figure 3: Survival function of OGE=IED

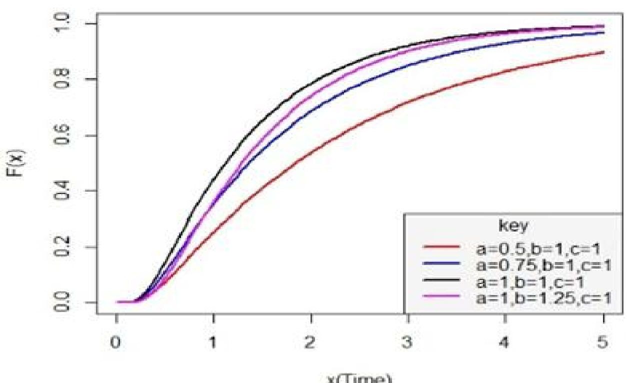

Figure 2: CDF plot of OGE-IED

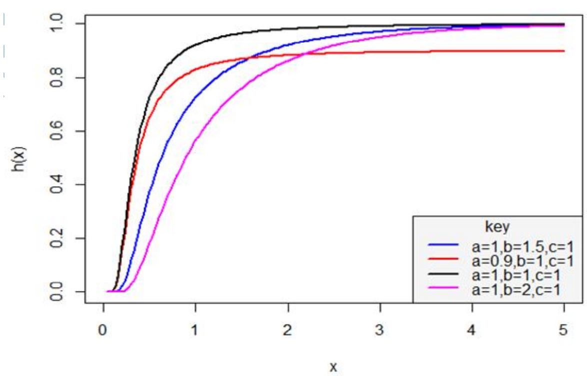

Figure 4: Hazard function of OGE-IED

Figures: 1, 2, 3 & 4: Figures 1, 2, 3 and 4 are the graphs of some of the possible shapes of the pdf, CDF, survival function and hazard function of OGE-IED for some values of the parameters $\boldsymbol{a} = \boldsymbol{\alpha}, \boldsymbol{b} = \boldsymbol{\beta}$ and $\boldsymbol{c} = \boldsymbol{\lambda}$ respectively.