Ratio and Product Estimators: New Strategies with Unknown Weight for finite Population

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***Abstract*—In this paper, ratio and product estimators with unknown weight for finite population mean under simple random sampling scheme are suggested. The properties (Biases and MSEs) of the proposed estimators are derived up to first order approximation. The expressions for the unknown weight α which optimized the efficiencies of the proposed estimators have been established. Theoretical and empirical studies have been done to demonstrate the efficiencies of the proposed estimators over traditional estimator. The results show that the proposed estimators are more efficient than conventional estimators considered in the study if their respective if their respective conditions in section 5 are satisfied.**

Keywords: *Auxiliary variable, Ratio estimator, Product estimator, Precision, Bias, Mean Squared Error.*

I. Introduction

The use of auxiliary information at the estimation stage helps in improving the precision of estimates of the population parameters. Ratio, product and regression methods of estimation are good examples in this context. If the correlation between study variate y and the auxiliary variate x is positive (high), the ratio method of estimation envisaged by Cochran [3] is used. On the other hand if the correlation between y (study variate) and x (auxiliary variate) is negative (high), the product method of estimation envisaged by Robson [13] and revisited by Murthy [10] can be employed quite effectively. Owing to this limitation of classical ratio and product estimators in recent past, effort have been made by several authors to develop more and more estimators which are either ratio or product type in nature but have lesser variance than the classical estimators. Some of these authors include Adewara et al. [1], Srivenkataramana and Srinath [17], Singh and Tailor [15], Sisodia and Dwivedi [16], Prasad [12], Koyuncu and Kadilar [9], Khoshnevisan, et. al.[8], Kadilar and Cingi [7].

The improvement and modification of ratio estimators were based on the use of known value of some population parameters like coefficient of variation, kurtosis, correlation, variance e.t.c. and use of linear transformations like Adewara et al. [1], Singh and Tailor[15], Sisodia and Dwivedi [16], Khoshnevisan, et. al. [8], e.t.c.

Also, some authors including Goodman [6], Srivastava[20], Ray and Sahai [14], Srivastava et al. [21], Pandey and Dubey [11], Chaubey et al. [4], Tripathy and Singh [23], Dubey [5], Bandyopadhyay [2], Tracy et al. [22], Singh et al. [18], Singh and Vishwakarma [19], etc. have suggested a variety of product type estimators and discussed their properties.

In this paper, ratio and product estimators with unknown weight for population mean of study variable y using information on an auxiliary variable x have been suggested and the conditions for their efficiencies have been established.

II. RESEARCH METHODOLOGY

Classical Estimators

Consider a population of N units associated with the i[th] units are the variables of main interest, Y_i and the auxiliary variables X_i for i= 1,2,3........N. Let $\bar{X}$ and $\bar{Y}$ be the population means of variables X and Y respectively. The coefficients of variation of these variables are denoted by $C_x^2 = S_x^2 / \bar{X}^2$ and $C_y^2 = S_y^2 / \bar{Y}^2$ respectively while S_y^2 and S_x^2 are their corresponding variances. Also, ρ will represent population correlation coefficients between X and Y respectively.

The classical ratio and product estimators of population mean are defined as

$$\bar{y}_R = \bar{y}\frac{\bar{X}}{\bar{x}} \qquad (1)$$

$$\bar{y}_p = \bar{y}\frac{\bar{x}}{\bar{X}} \qquad (2)$$

The Bias and Mean Square Error (MSE) of the equations (1) and (2) are respectively given as

$$Bias(\bar{y}_R) = \bar{Y}\theta(C_x^2 - \rho C_y C_x) \qquad (3)$$

$$MSE(\bar{y}_R) = \bar{Y}^2\theta(C_y^2 + C_x^2 - 2\rho C_x C_y) \qquad (4)$$

$$Bias(\bar{y}_p) = \bar{Y}\theta\rho C_y C_x \qquad (5)$$

$$MSE(\bar{y}_p) = \bar{Y}^2\theta(C_y^2 + C_x^2 + 2\rho C_x C_y) \qquad (6)$$

Suggested Class of Estimators

Motivated by Adewara et. al. [1], the following estimators are proposed.

$$\bar{y}_{\sqrt{R1}} = \bar{y}\frac{\sqrt{\bar{X}}}{\sqrt{\bar{X}}+\alpha(\sqrt{\bar{x}}-\sqrt{\bar{X}})} \qquad (7)$$

$$\bar{y}_{p1} = \bar{y}\frac{\alpha\bar{x}}{\alpha\bar{X}+(1-\alpha)(\bar{x}-\bar{X})} \qquad (8)$$

where $\alpha(0 < \alpha < 1)$is to be determined so as to minimize the MSEs of proposed strategies.

In order to study the large sample model based properties of the proposed estimators, we define sample means as,

$$\bar{y} = \bar{Y}(1+e_0) \text{ and } \bar{x} = X(1+e_1) \qquad (9)$$

such that

$$|e_i| < 1,\ E(e_i) = 0,\ \ E(e_0^2) = \frac{\theta\mu_{20}}{\bar{Y}^2} = \theta C_y^2,\ E(e_1^2) = \frac{\theta\mu_{02}}{\bar{X}^2} = \theta C_x^2,$$
$$E(e_0 e_1) = \frac{\theta\mu_{11}}{\bar{Y}\bar{X}} = \theta\rho C_y C_x \qquad (10)$$

where

$$f = \frac{n}{N},\ \theta = \frac{1-f}{n} \text{ and } \mu_{rs} = E(y-\bar{Y})^r(x-\bar{X})^s$$

Expressing equations (7), and (8) in terms of e's to second degree terms of order n^{-1}, we have

$$\bar{y}_{\sqrt{R1}} - \bar{Y} = \bar{Y}\left(e_0 - \frac{1}{2}\alpha\left(e_1 - \frac{1}{4}e_1^2 + e_0e_1\right) + \frac{\alpha^2}{4}e_1^2\right) \qquad (11)$$

$$\bar{y}_{p1} - \bar{Y} = \bar{Y}(e_0 - \frac{1-2\alpha}{\alpha}e_1 + \frac{(1-\alpha)(1-2\alpha)}{\alpha^2}e_1^2 - \frac{1-2\alpha}{\alpha}e_0e_1) \qquad (12)$$

Bias And Mean Square Error (MSE) of The Suggested Estimators

The biases of the proposed estimators can be obtained by taking the expectations of equations (11) and (12) and using results in (10) as;

$$Bias(\bar{y}_{\sqrt{R1}}) = \bar{Y}\theta\left(\frac{\alpha}{2}\left(\frac{1}{4}C_x^2 - \rho C_y C_x\right) + \frac{\alpha^2}{4}C_x^2\right) \qquad (13)$$

$$Bias(\bar{y}_{p1}) = \bar{Y}\theta(\frac{(1-\alpha)(1-2\alpha)}{\alpha^2}C_x^2 - \frac{1-2\alpha}{\alpha}\rho C_y C_x) \qquad (14)$$

Squaring both sides of equations (11) and (12) then taking expectations and using results in (10), we obtain the MSE of the proposed estimators to terms of order n^{-1} as

$$MSE(\bar{y}_{\sqrt{R1}}) = \bar{Y}^2\theta(C_y^2 + (\tfrac{\alpha}{2})^2C_x^2 - \alpha\rho C_x C_y) \qquad (15)$$

$$MSE(\bar{y}_{p4}) = \bar{Y}^2\theta(C_y^2 + (\tfrac{1-2\alpha}{\alpha})^2C_x^2 - 2\frac{1-2\alpha}{\alpha}\rho C_x C_y) \qquad (16)$$

Minimization of equations (15) and (16) with respect to α yield their optimum values whenα values are given respectively as;

$$\alpha_{\sqrt{R1}} = \frac{2\rho C_y}{C_x}, \qquad \alpha_{p1} = \frac{C_x}{2C_x + \rho C_y}$$

Efficiency Comparisons

Here, efficiencies of the proposed estimators are compared with that traditional estimator with condition that $0 < \rho < 1$ and $0 < \alpha < 1$.

(i) $MSE(\bar{y}_R) - MSE(\bar{y}_{\sqrt{R1}}) = \bar{Y}^2\theta(\left(1 - \frac{\alpha^2}{4}\right)C_x^2 + (\alpha - 2)\rho C_x C_y)$

which is positive if$max\left(0, \frac{4\rho C_y - 2C_x}{C_x}\right) < \alpha < 1$

(ii) $MSE(\bar{y}_p) - MSE(\bar{y}_{p1}) = \bar{Y}^2\theta(\left(1-\left(\frac{1-2\alpha}{\alpha}\right)^2\right)C_x^2 + 2(1+\frac{1-2\alpha}{\alpha})\rho C_x C_y)$ Which is positive if$max\left(0, \frac{C_x}{3C_x+2\rho C_y}\right) < \alpha < 1$

III. Empirical Study And Results

The MSEs and biases of the suggested estimators over ranges of the unknown weight are demonstrated using population parameters. The Descriptions of the population parameters are given in Table 1 and Table 2.

Table 1: Description and parameters of the populations

Parameters	N	n	ρ	R	$\bar{X}$	$\bar{Y}$	C_x^2	C_y^2
Values	21	5	0.974	2.222	1.714	3.810	0.523	0.116

Source: Cochran 1977, page 186 , Y: Total no. of Members, X: No. of Children

Table 2: Descriptionof the population's parameters

Parameters	N	n	ρ	R	$\bar{X}$	$\bar{Y}$	C_x^2	C_y^2
Values	80	5	-0.801	0.283656	118.1625	33.5175	0.230835	0.082754

Source: US Environmental Protection Agency 1991, Y: Average miles per gallon, X: Engine horsepower

Table 3: Biases of the Suggested Estimators

Estimator	**Ranges of Weight α**								
	0.1	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**
$\bar{y}_{\sqrt{R1}}$	-0.002	-0.003	-0.003	0.000	0.003	0.008	0.015	0.023	0.033
$\bar{y}_{p1}$	110.016	19.496	5.441	1.436	0.0	-0.554	-0.753	-0.794	-0.762

$Bias(\bar{y}_R) = 0.164$, $Bias(\bar{y}_R) = -0.696$

Table 3 shows the biases of the proposed estimators at different values of $\boldsymbol{\alpha}$**.** it is observed that the estimator $\bar{y}_{\sqrt{R1}}$ is almost unbiased at all values of α.The estimator $\bar{y}_{p1}$ is unbiased at $\alpha = 0.5$ and the effective ranges for which is less bias than $\bar{y}_p$ is$0.5 < \alpha < 0.6$.

Table 4: PRE of Suggested Estimators over Traditional Estimators

Estimator	**Ranges of Weight α**									**Optimal value of α**
	0.1	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
$\bar{y}_{\sqrt{R1}}$	169.6	216.0	283.5	385.9	549.6	824.8	1298.8	2032.2	2639.6	0.92
$\bar{y}_{p1}$	0.55	3.26	11.69	36.70	111.39	266.41	291.66	198.06	134.90	0.658

Table 4 shows the percent relative efficiencies of the proposed estimators over traditional ratio and product estimators at different values of $\boldsymbol{\alpha}$. It is observed that the estimator $\bar{y}_{\sqrt{R1}}$is more efficient than traditional estimator $\bar{y}_R$ at all values of α. The effective ranges of$\boldsymbol{\alpha}$that make estimators$\bar{y}_{p1}$more efficient than traditional estimator $\bar{y}_p$ is $0.5 < \alpha < 0.9$. Table 4 also shows optimum values of α for which the suggested estimators attained maximum precision for the data used.

IV. Conclusion

From efficiency comparisons in Section 5 and the empirical results of the percentage relative efficiency of the proposed estimators in section 6, the proposed of ratio and product estimators are more efficient than traditional estimator $\bar{y}_R$and $\bar{y}_p$under the stipulated conditions and can produce better estimates if their respective optimum values of unknown weight are utilized.

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